CHRISTMAS MATHEMATICS COMPETITIONS

American Mathematics Competitions

The problems and solutions for this AMC 12 were prepared by the problem writing Committee for the mock AMC 10 and AMC 12:

atmchallenge

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Th3Numb3rTh33

tree3

MOCK 2018 AIME

The 1st annual AIME will be released on Friday, December 22, with the alternate on Friday, December 29. It is a 15-question, 3-hour, integer-answer exam. All students are invited to participate. All students are also invited to take the USA Mathematical Olympiad (USAMO) which will be released on January 5, 2017. The best way to prepare for the AIME and USAMO is to study previous exams. Copies may be ordered as indicated below.

PUBLICATIONS

A complete listing of current publications, with ordering instructions, is at the MAA web site:
maa.org/math-competitions

CHRISTMAS MATHEMATICS COMPETITIONS

American Mathematics Competitions

1st Annual

AMC 12 A

American Mathematics Contest 12 A

FRIDAY, December 22, 2017

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR TELLS YOU.
2. This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. Mark your answer to each problem on the AMC 12 Answer Form with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded.
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than scratch paper, graph paper, rulers, compass, protractors, and erasers. No calculators are allowed. No problems on the test will *require* the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your proctor will ask you to record certain information on the answer form.
8. When your proctor gives the signal, begin working on the problems. You will have **75 minutes** to complete the test.
9. When you finish the exam, *sign your name* in the space provided on the Answer Form.

The Committee on the American Mathematics Competitions (CAMC) reserves the right to re-examine students before deciding whether to grant official status to their scores. The CAMC also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

All students will be invited to take the 1st annual American Invitational Mathematics Examination (AIME) released on Friday, December 22, 2017 or Friday, December 29, 2017. More details about the AIME and other information are on the back page of this test booklet.

1. What is the value of $(2+0+1)^7 - 201 + 7 \times (2+0) + 17$?

(A) 559 **(B)** 1983 **(C)** 1989 **(D)** 2008 **(E)** 2017

2. There exists unique digits $a \neq 0$ and $b \neq a$ such that the four-digit number $\overline{aabb}$ is a perfect square. Compute $a+b$.

(A) 4 **(B)** 7 **(C)** 8 **(D)** 11 **(E)** 14

3. Let (x, y) be a pair of real numbers satisfying $4x^2 - 6xy + 9y^2 = 2016$. Among all possible pairs, compute the positive difference between the maximum and minimum possible values of xy.

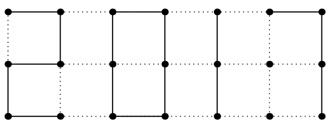
(A) 0 **(B)** 112 **(C)** 224 **(D)** 336 **(E)** 448

4. How many ordered pairs (x, y) of integers satisfy

$$x^4 + y^4 + 2x^2y^2 + 1023 = 1024x^2 + 1024y^2 \text{ ?}$$

(A) 4 **(B)** 8 **(C)** 12 **(D)** 16 **(E)** 20

5. The distance between two adjacent points on the grid below is 1. What is the minimum number of dotted line segments of length 1 that need to be replaced with black line segments so that at least 21 different rectangles can be traced along the black line segments?



(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

6. Call a positive integer *super-even* if the last two digits of all its multiples are always even. If S is the sum of all super-even integers $1 \leq n \leq 2018$, what is the number of factors of S?

(A) 12 **(B)** 18 **(C)** 24 **(D)** 32 **(E)** 36

7. Let p, q, and r be distinct prime numbers such that p^2+1, q^2+2, and r^2+3, form an arithmetic sequence in that order. What is the minimum possible value of $p+q+r$?

(A) 15 **(B)** 25 **(C)** 30 **(D)** 37 **(E)** 40

2018
AMC 12 A

DO NOT OPEN UNTIL FRIDAY, December 22, 2017

****Administration On An Earlier Date Will Disqualify Your Schools Results****

1. All information (Rules and Instructions) needed to administer this exam is contained in the TEACHERS MANUAL, which is outside of this package. PLEASE READ THE MANUAL BEFORE DECEMBER 22, 2017. Nothing is needed from inside this package until DECEMBER 22.
2. Your PRINCIPAL or VICE-PRINCIPAL must verify on the AMC 10 CERTIFICATION FORM (found in the Teachers Manual) that you followed all rules associated with the conduct of the exam.
3. The Answer Forms must be submitted through the Google Form provided no later than December 29.
4. *The publication, reproduction or communication of the problems or solutions of this test during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet or media of any type is a violation of the competition rules.*

The
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22. Let S be the set of all points (a, b, c) in 3-dimensional space with all coordinates nonnegative, such that $ax^2 + bx + c = 2016$ has real solutions in the interval $m \le x \le n$ for positive real numbers m and n. If $n - m = 2mn = 16$, then the volume of S can be written as $2016^3 \cdot R$ for some real number R. What is R?

(A) $\frac{35}{24}$ **(B)** $\frac{8}{3}$ **(C)** $2\sqrt{2}$ **(D)** $\frac{35}{4}$ **(E)** $12\sqrt{2}$

23. For how many positive integers $n < 2018$ is the number of terms in the expansion of

$$\left(x^{45}+x^{42}+x^{40}\right)^n+\left(x^{20}+x^{18}+x^{15}\right)^{2n}+\left(x^{10}+x^{7}+x^{4}\right)^{3n}+\left(x^{3}+x^{2}+x\right)^{4n}$$

n less than a perfect square?

(A) 0 **(B)** 1 **(C)** 3 **(D)** 5 **(E)** 8

24. Let $P_0, P_1, \ldots$ be points in the complex plane such that $P_0 = -1, P_1 = i$ and $P_n = i \cdot P_{n-1} + P_{n-2}$ for every $n \ge 2$. Let $\mathcal{P}$ be the polygon $P_0P_1 \ldots P_{2017}$ and let P and A be its perimeter and area, respectively. If the value of $\frac{P}{A}$ can be written in the form $\frac{a\sqrt{b}+c\sqrt{d}}{e}$ for positive integers a, b, c, d, and e such that b and d are not divisible by the square of any prime and $\gcd(a, c, e) = 1$, compute the remainder when $a + b + c + d + e$ is divided by 100.

(A) 8 **(B)** 12 **(C)** 16 **(D)** 20 **(E)** 24

25. A group of 5 students sit around a circle. Each minute, each person points at another student and that person is thus removed from the circle. What is the expected number of minutes it will take until either a single person or nobody remains? (Note that two people could be pointing at the same person)

(A) $\frac{35}{27}$ **(B)** $\frac{99}{64}$ **(C)** $\frac{175}{108}$ **(D)** $\frac{415}{256}$ **(E)** $\frac{140}{81}$

8. Two random, not necessarily distinct, permutations of the digits 2017 are selected and added together. What is the expected value of this sum?

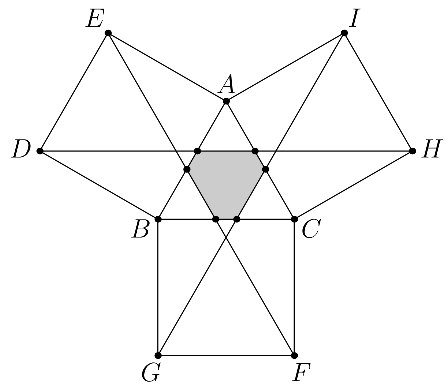
(A) 3737 **(B)** 3840 **(C)** 4034 **(D)** 5200 **(E)** 5555

9. Three triplets, Tam, Tom, and Tim, work together to build a treehouse. Tam works twice as fast as Tom, but three times slower than Tim. After working together for 10 days, Tim decides to quit since he is tired. After another 10 days, Tom also decides to quit. It takes Tam another 6 days to complete the treehouse. The number of days it would had taken them to complete the treehouse if none of them had quit can be written as $\frac{m}{n}$ for relatively prime positive integers m and n. What is $m + n$?

(A) 22 **(B)** 47 **(C)** 53 **(D)** 66 **(E)** 91

10. Equilateral triangle $\triangle ABC$ has $AB = BC = AC = 1$. Squares $ABDE$, $BCFG$, and $ACHI$ are constructed outside $\triangle ABC$ with sides coinciding with AB, BC, and AC, respectively. Connect EF, GI, and DH, so that they intersect $\triangle ABC$ at six points, forming a hexagon. Then, the area of the hexagon can be expressed in the form $\frac{a-b\sqrt{c}}{d}$ for some positive integers a, b, c, d, where c is not divisible by the square of any prime and $\gcd(a, b, d) = 1$. Compute $a + b + c + d$.

(A) 10 **(B)** 12 **(C)** 16 **(D)** 22 **(E)** 26



11. For positive integers p and q, let $\boxed{\text{pq}} = (pq + p + q)(p - q)$. Compute $\boxed{10} + \boxed{11} + \cdots + \boxed{98} + \boxed{99}$.

(A) 255 **(B)** 265 **(C)** 275 **(D)** 285 **(E)** 295

12. What is the sum of the digits of the smallest prime factor of $712! + 1$?

(A) 4 **(B)** 5 **(C)** 9 **(D)** 11 **(E)** 17

13. In circle ω, with radius $3\sqrt{2}$, let A, B, C, and D be points on ω such that $ABCD$ is a square. Draw four regular hexagons such that AB, BC, CD, and DA are opposite vertices of the hexagons. Then there exists a circle that passes through exactly two vertices of each hexagon. If the largest possible area of this circle can be written as $(a + b\sqrt{c})\pi$, where a, b, c are integers and c is not divisible by the square of any prime, find $a + b + c$.

(A) 18 **(B)** 21 **(C)** 24 **(D)** 27 **(E)** 30

14. Consider all positive integers n, with the property that $35 \times n$ has exactly 3500 digits. Let N be the largest integer with such property. What is the sum of the digits of N?

(A) 15743 **(B)** 15751 **(C)** 15752 **(D)** 15753 **(E)** 15761

15. In $\triangle ABC$, $AB = 8, BC = 13$, and $AC = 15$. The line passing through A and the circumcenter of $\triangle ABC$ intersects the circumcircle at a point $P \neq A$. Compute $AP^2 + BP^2 + CP^2$.

(A) 383 **(B)** 384 **(C)** 385 **(D)** 386 **(E)** 387

16. Define $f(n)$ to be the sum of the positive integer factors of n. If $f(n) = 360$ and $f(3n) = 1170$, what is the smallest possible value of $f(2n)$?

(A) 720 **(B)** 729 **(C)** 744 **(D)** 756 **(E)** 840

17. Eddie has a birthday party, in which he invited his 8, very generous, best friends. Each friend brought a certain number of gifts for Eddie. It is observed that for every group of 4 friends, at least 2 of them brought the same number of gifts. Moreover, the person with the most gifts brought 4, while the least brought 1. What is the total number of possible values of the amount of gifts Eddie received in total?

(A) 19 **(B)** 20 **(C)** 21 **(D)** 22 **(E)** 23

18. Let $p(x)$ and $q(x)$ be two distinct polynomials with integer coefficients, of degree less or equal to 6, that satisfy $p(k) = q(k)$ and $p(-k) = -q(-k)$ for all integers $1 \leq k \leq 6$. What is the number of positive divisors of the minimum value of $p(0)^2 + q(0)^2$?

(A) 30 **(B)** 75 **(C)** 108 **(D)** 135 **(E)** 192

19. In $\triangle ABC$, $AB = 4$, $BC = 5$, $AC = 6$, and I is the incenter of the triangle. Let $\mathcal{S}$ be the locus of all points D such that the circumcenter O of $\triangle ABD$ is collinear with AI. Determine the largest integer less than or equal to the perimeter of $\mathcal{S}$.

(A) 10 **(B)** 12 **(C)** 14 **(D)** 16 **(E)** 18

20. A lattice point is a point in the coordinate plane, whose coordinates are both integers. There exists exactly 2017 lattice points that lie inside or on the region enclosed by the lines $x = 0, y = 0$, and $3x + 4y = 4a$, for some positive integer a. How many of these lattice points lie strictly inside this region?

(A) 1873 **(B)** 1909 **(C)** 1921 **(D)** 1945 **(E)** 1963

21. In $\triangle ABC$, let the bisector of $\angle BAC$ meet BC and the circumcircle of $\triangle ABC$ at points D and E, respectively, and let F be the midpoint of BC. If $[DEF] = 1/3$, $DF = 1/2$, and $BC = 5$, what is the area of $\triangle ABC$?

(A) $\frac{197}{20}$ **(B)** $\frac{720}{73}$ **(C)** $\frac{15\sqrt{7}}{4}$ **(D)** $\frac{289}{27}$ **(E)** $\frac{16\sqrt{7}}{3}$