.... *Christmas Math Competitions*

CMC ARML 2020

Solutions Document

Albert Wang, Allen Baranov, Andrew Wen, Ankan Bhattacharya, Anthony Wang, Eric Shen, Elliott Liu, Federico Clerici, Joseph Zhang, Justin Lee, Kaiwen Li, Kyle Lee, Luke Choi, Mason Fang, Minjae Kwon, Nathan Xiong, Preston Fu, Raymond Feng, Rishabh Das, Sean Li, Tovi Wen, and Valentio Iverson

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Contents

§1 Individual problems

1. Compute the maximum value of n for which n cards, numbered 1 through n, can be arranged and lined up in a row such that

- it is possible to remove 20 cards from the original arrangement leaving the remaining cards in ascending order, and
- it is possible to remove 20 cards from the original arrangement leaving the remaining cards in descending order.

2. Let $ABCD$ be a quadrilateral with side lengths $AB = 2$, $BC = 5$, $CD = 3$, and suppose $\angle B = \angle C = 90°$. Let M be the midpoint of $\overline{AD}$ and let P be a point on $\overline{BC}$ so that quadrilaterals $ABPM$ and $DCPM$ have equal areas. Compute PM.

3. There is a unique nondecreasing sequence of positive integers a_1, a_2, …, a_n such that

$$\left(a_1 + \frac{1}{a_1}\right)\left(a_2 + \frac{1}{a_2}\right)\cdots\left(a_n + \frac{1}{a_n}\right) = 2020.$$

Compute $a_1 + a_2 + \cdots + a_n$.

4. Let $0° < \theta < 90°$ be an angle. If

$$\log_{\sin\theta}\cos\theta, \quad \log_{\cos\theta}\tan\theta, \quad \log_{\tan\theta}\sin\theta$$

form a geometric progression in that order, compute $\sin\theta$.

5. Let ABC be a triangle and let M be the midpoint of $\overline{BC}$. The lengths AB, AM, AC form a geometric sequence in that order. The side lengths of $\triangle ABC$ are 2020, 2021, x in some order. Compute the sum of all possible values of x.

6. Let $\mathcal{C} = \{(x, y, z) : 0 \le x, y, z \le 1\}$. Real numbers a, b, c are selected randomly and independently such that $0 < a, b, c < 1$. Given that $\mathcal{C}$ and the plane $ax + by + cz = 1$ intersect, compute the probability that their intersection is a nondegenerate hexagon.

7. Compute

$$21\left(1 + \frac{20}{2}\left(1 + \frac{19}{3}\left(1 + \frac{18}{4}\left(\cdots\left(1 + \frac{12}{10}\right)\cdots\right)\right)\right)\right).$$

8. Let ABC be an equilateral triangle with circumcircle ω. Select a point P on the minor arc BC of ω such that the distance from P to line AB is 1, and so that the distance from P to line AC is 2. Compute the side length of $\triangle ABC$.

9. Let $\lceil x \rceil$ denote the smallest integer greater than or equal to x. The sequence (a_i) is defined as follows: $a_1 = 1$, and for all $i \ge 1$,

$$a_{i+1} = \min\left\{7\left\lceil\frac{a_i + 1}{7}\right\rceil, 19\left\lceil\frac{a_i + 1}{19}\right\rceil\right\}.$$

Compute a_{100}.

10. Let $S(n)$ denote the sum of the digits of a positive integer n. Compute the number of positive integers n for which

- all of n's digits are nonzero, and
- $(S(2n))^2 + 2S(n) + 1 = 345$.

§2 Individual statistics

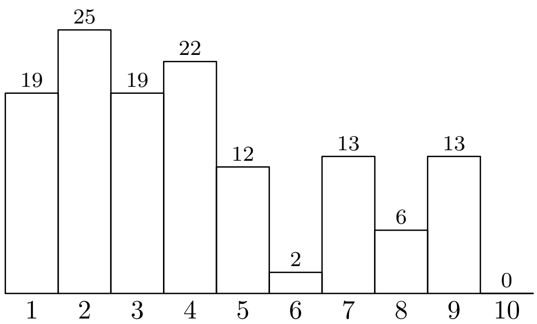
§2.1 Leaderboard

Note: Contestants with scores of 7 or above were invited to participate in the tiebreaker round. Ties were broken as follows, in decreasing level of precedence: (i) raw individual round score; (ii) tiebreaker score, if applicable; (iii) problems solved; (iv) time of submission.

Username	Score
kvedula2004	8
—	8
dchenmathcounts	7
MP8148	7
—	7
Archeon	7
Stormersyle	6
KaiDaMagical336	6
kevinmathz	6
superagh	5
cmsgr8er	5
mathtiger6	5
—	5
kred9	5
onezero	5
CT17	5
a.y.711	5
os31415	4
lrjr24	4
bissue	4
—	3
will3145	3
GerpTheGreat	3
Mathletesv	2
v4913	2
usernameyourself	2
GammaZero	1
—	1

§2.2 Solve-rates

A total of 28 contestants participated in the individual round.



§2.3 Answer Key

I-1. 41

I-2. $\dfrac{\sqrt{26}}{2}$

I-3. 19

I-4. $\dfrac{\sqrt{5}-1}{2}$

I-5. $8082+\sqrt{2}$

I-6. $\dfrac{3}{10}$

I-7. $2^{20}-1$ (or 1048575)

I-8. $\dfrac{14}{9}\sqrt{3}$

I-9. 525

I-10. 6600

§3 Individual solutions

§3.1 Problem I1, by Justin Lee

Compute the maximum value of n for which n cards, numbered 1 through n, can be arranged and lined up in a row such that

- it is possible to remove 20 cards from the original arrangement leaving the remaining cards in ascending order, and
- it is possible to remove 20 cards from the original arrangement leaving the remaining cards in descending order.

The answer is 41, achieved by

$$21,\ 20,\ 19,\ \dots,\ 1,\ 22,\ 23,\ \dots,\ 41.$$

We now prove the upper bound.

Let A be the set of 20 cards that, upon their removal, results in an ascending arrangement; equivalently let B be that which results in a descending arrangement.

Evidently $\overline{A \cup B}$ is both ascending and descending, so it has size at most 1. Then

$$n \le |A| + |B| + |\overline{A \cup B}| \le 41,$$

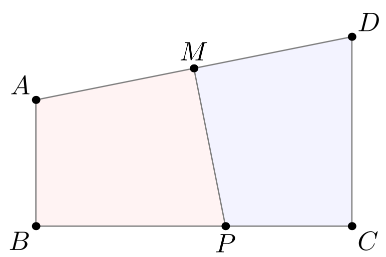
as needed.

§3.2 Problem I2, by Kyle Lee

Let $ABCD$ be a quadrilateral with side lengths $AB = 2$, $BC = 5$, $CD = 3$, and suppose $\angle B = \angle C = 90^\circ$. Let M be the midpoint of $\overline{AD}$ and let P be a point on $\overline{BC}$ so that quadrilaterals $ABPM$ and $DCPM$ have equal areas. Compute PM.

We present three solutions. There is also a direct solution by shoelace formula.

First solution, by congruence Consider the point P on $\overline{BC}$ so that $BP = 3$, $CP = 2$.



Then $\triangle ABP \cong \triangle PCD$, so $PA = PD$. But $MA = MD$, so $\overline{PM}$ is the perpendicular bisector of $\overline{AD}$. It follows that $ABPM$ is cyclic, so $\angle MAB = \angle MPC$ and $\angle MPB = \angle MDC$.

It is clear $ABPM$ and $PCDM$ are congruent cyclic quadrilaterals, so they have the same area. Furthermore $MA = MP = MC$, so $MP = AD/2 = \sqrt{26}/2$.

Second solution, by area chasing The area of $ABCD$ is $25/2$, so the area of $ABPM$ is $25/4$. Let N be the midpoint of $\overline{BC}$, so $\overline{MN} \perp \overline{BC}$. Then $MN = NB = 5/2$, so the area of $ABNM$ is $45/8$. Then the area of $\triangle MNP$ is $5/8$, so $NP = 1/2$. By Pythagorean theorem on $\triangle MNP$, we have $MP = \sqrt{26}/2$.

Third solution, for general trapezoids For any trapezoid $ABCD$ with $\overline{AB} \parallel \overline{CD}$ and $AB : CD = 2 : 3$, I claim the point P on $\overline{BC}$ with $BP : CP = 3 : 2$ satisfies $[ABPM] = [ACPM]$. (In what follows, $[\mathcal{P}]$ denotes the area of $\mathcal{P}$.)

Let $T = \overline{AD} \cap \overline{BC}$. Let $AD = 2x$, $BC = 2y$, so that $TA = 4x$, $TB = 4y$. Also let $TP = zy$. We have

$$\frac{[TAB]}{[TMP]} = \frac{TA \cdot TB}{TM \cdot TP} = \frac{16}{5z} \quad \text{and} \quad \frac{[TCD]}{[TMP]} = \frac{TC \cdot TD}{TM \cdot TP} = \frac{36}{5z}.$$

By hypothesis, $2[TMP] = [TAB] + [TCD]$, so $z = 26/5$.

It follows that $BP = 3$, $CP = 2$. If N is the midpoint of $\overline{BC}$, then $\overline{MN} \perp \overline{BC}$, $MN = 5/2$, $NP = 1/2$, so $MP = \sqrt{26}/2$.

§3.3 Problem I3, by Minjae Kwon

There is a unique nondecreasing sequence of positive integers a_1, a_2, …, a_n such that

$$\left(a_1 + \frac{1}{a_1}\right)\left(a_2 + \frac{1}{a_2}\right) \cdots \left(a_n + \frac{1}{a_n}\right) = 2020.$$

Compute $a_1 + a_2 + \cdots + a_n$.

We can express each of 2020's prime factors:

$$\begin{aligned} 2 &= \left(1 + \frac{1}{1}\right) \\ 5 &= \left(1 + \frac{1}{1}\right)\left(2 + \frac{1}{2}\right) \\ 101 &= \left(1 + \frac{1}{1}\right)^2 \left(2 + \frac{1}{2}\right)\left(10 + \frac{1}{10}\right). \end{aligned}$$

It follows that

$$2020 = 2^2 \cdot 5 \cdot 101 = \left(1 + \frac{1}{1}\right)^5 \left(2 + \frac{1}{2}\right)^2 \left(10 + \frac{1}{10}\right),$$

so $a_1 + \cdots + a_n = 5 \cdot 1 + 2 \cdot 2 + 1 \cdot 10 = 19$.

§3.4 Problem I4, by Tovi Wen and Raymond Feng

Let $0^\circ < \theta < 90^\circ$ be an angle. If

$$\log_{\sin\theta} \cos\theta, \quad \log_{\cos\theta} \tan\theta, \quad \log_{\tan\theta} \sin\theta$$

form a geometric progression in that order, compute $\sin\theta$.

The product of the three terms is 1, so $\log_{\cos\theta} \tan\theta = 1$, or $\cos\theta = \tan\theta$. It follows that $\sin\theta = \cos^2\theta = 1 - \sin^2\theta$, so $\sin\theta = \frac{\sqrt{5}-1}{2}$.

§3.5 Problem I5, by Eric Shen

Let ABC be a triangle and let M be the midpoint of $\overline{BC}$. The lengths AB, AM, AC form a geometric sequence in that order. The side lengths of $\triangle ABC$ are 2020, 2021, x in some order. Compute the sum of all possible values of x.

The key claim is this:

Claim. $AM^2 = AB \cdot AC$ if and only if $BC = |AB - AC|\sqrt{2}$

Proof. The proof is by the Median formula

$$AM^2 = \frac{2\left(AB^2 + AC^2\right) - BC^2}{4},$$

which is a corollary of Stewart's theorem. The above equation rearranges to

$$\frac{BC^2}{2} = AB^2 + AC^2 - 2AM^2,$$

and the claim follows. $\square$

Evidently all triangle satisfying $BC = |AB - AC|\sqrt{2}$ obey the triangle inequality. Using symmetry in B, C, we take cases:

- if $AB = 2020$, $AC = 2021$, then $BC = \sqrt{2}$.
- if $AB = 2020$, $BC = 2021$, then $AC = 2020 \pm 1010.5\sqrt{2}$.
- if $AB = 2021$, $BC = 2020$, then $AC = 2021 \pm 1010\sqrt{2}$.

Grouping conjugate pairs, the answer is $\sqrt{2} + 2 \cdot 2020 + 2 \cdot 2021 = 8082 + \sqrt{2}$.

§3.6 Problem I6, by Justin Lee

Let $\mathcal{C} = \{(x, y, z) : 0 \le x, y, z \le 1\}$. Real numbers a, b, c are selected randomly and independently such that $0 < a, b, c < 1$. Given that $\mathcal{C}$ and the plane $ax + by + cz = 1$ intersect, compute the probability that their intersection is a nondegenerate hexagon.

The probability $\mathcal{C}$ and $ax + by + cz = 1$ intersect is the probability $(1, 1, 1)$ lies above $\mathcal{C}$; i.e. $a + b + c \ge 1$, which occurs with probability $5/6$. It remains to show the probability the intersection is a hexagon is $1/4$, thence the answer is $3/10$.

Observe that $(0, 0, 0)$, $(0, 0, 1)$, $(0, 1, 0)$, $(1, 0, 0)$ are always below the plane; that is, $ax + by + cz \le 1$. It is clear the intersection of $\mathcal{C}$ and $ax + by + cz = 1$ is a hexagon if and only if the other four vertices are below the plane, so $a + b > 1$, $b + c > 1$, $c + a > 1$.

First finish, geometrically The intersection of $a + b > 1$, $b + c > 1$, $c + a > 1$ on the abc-plane is a triangular bipyramid with vertices $(1/2, 1/2, 1/2)$, $(1, 1, 1)$ and whose base has vertices $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$. The base has area $\sqrt{3}/2$ and the height of the bipyramid is $\sqrt{3}/2$, so the volume of the region is $1/4$.

Second finish, by calculus For $a < 1/2$, the union of $b + c > 1$, $b > 1 - a$, $c > 1 - a$ on the bc-plane is the upper-right square of side length a, and when $a < 1/2$, it is the same square of side length a minus an isosceles right triangle of side length $2a - 1$. Hence the area of the region is

$$a^2 - \frac{(2a-1)^2}{2} = -a^2 + 2a - \frac{1}{2}.$$

Hence, the volume of the region on the abc-plane is

$$\int_0^{1/2} a^2 \, da + \int_{1/2}^{1} \left(-a^2 + 2a - \frac{1}{2}\right) da = \frac{1}{24} + \frac{5}{24} = \frac{1}{4}.$$

§3.7 Problem I7, by Albert Wang

Compute

$$21\left(1+\frac{20}{2}\left(1+\frac{19}{3}\left(1+\frac{18}{4}\left(\cdots\left(1+\frac{12}{10}\right)\cdots\right)\right)\right)\right).$$

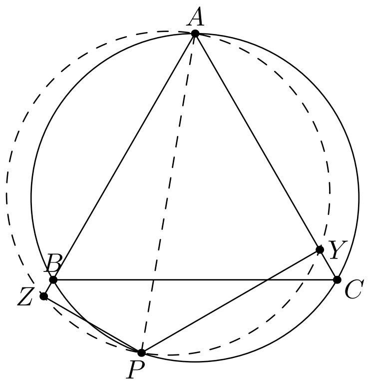
The expression equals

$$\frac{21}{1}+\frac{21}{1}\frac{20}{2}+\frac{21}{1}\frac{20}{2}\frac{19}{3}+\cdots+\frac{21\cdots 12}{1\cdots 10}=\binom{21}{1}+\binom{21}{2}+\cdots+\binom{21}{10}=2^{20}-1.$$

§3.8 Problem I8, by Tovi Wen

Let ABC be an equilateral triangle with circumcircle ω. Select a point P on the minor arc BC of ω such that the distance from P to line AB is 1, and so that the distance from P to line AC is 2. Compute the side length of $\triangle ABC$.

Let Y, Z be projections from P to $\overline{AC}$, $\overline{AB}$. The solution consists of four easy steps:



- P is the Miquel point of $BCYZ$, so $CP = 2BP$, $AP = 3BP$.
- $AYPZ$ is cyclic, so $\angle YPZ = 120^\circ$ and $YZ = \sqrt{7}$.
- Since $\overline{AP}$ is a diameter of (AYZ), we have $AP = 2\sqrt{7/3}$, i.e. $BP = 2\sqrt{21}/9$.
- Finally $BC = YZ \cdot PB/PZ = \sqrt{7}\cdot 2\sqrt{21}/9 = 14\sqrt{3}/9$

§3.9 Problem I9, by Justin Lee

Let $\lceil x \rceil$ denote the smallest integer greater than or equal to x. The sequence (a_i) is defined as follows: $a_1 = 1$, and for all $i \geq 1$,

$$a_{i+1} = \min\left\{7\left\lceil\frac{a_i+1}{7}\right\rceil, 19\left\lceil\frac{a_i+1}{19}\right\rceil\right\}.$$

Compute a_{100}.

Let $S = \{7, 14, 21, \dots\} \cup \{19, 38, 57, \dots\}$ be the set of multiples of either 7 or 19. Then the sequence $(a_i)_{i\geq 2}$ is the elements of S in increasing order.

We seek the 99th smallest element of S. But note that $7\cdot 19\cdot a$ is the $25a$th smallest element of S, so $a_{101} = 4\cdot 7\cdot 19 = 532$. It follows that $a_{100} = a_{101} - 7 = 525$.

§3.10 Problem I10, by Kyle Lee

Let $S(n)$ denote the sum of the digits of a positive integer n. Compute the number of positive integers n for which

- all of n's digits are nonzero, and
- $(S(2n))^2 + 2S(n) + 1 = 345$.

Indeed only a single pair $(S(n), S(2n))$ works. This follows from two orthogonal claims:

Claim 1. $S(2n) \in \{4, 10, 16\}$.

Proof. First $S(2n) \leq 18$ for size reasons.

Take the given equation modulo 9: we have $4n^2 + 2n + 1 \equiv 3 \pmod 9$. By modulo 3 analysis this should hold only for $n \equiv 2 \pmod 3$, and it is easy to check all $n \equiv 2, 5, 8 \pmod 9$ work.

Then $2n \equiv 1, 4, 7 \pmod 9$, and since $S(2n)$ must be even, $S(2n) \in \{4, 10, 16\}$. $\square$

Claim 2. $S(n) \leq 5S(2n)$.

Proof. In the multiplication $n \times 2$, we need at least $\frac{2S(n)-S(2n)}{9}$ carry-overs, which implies

$$S(n) \geq \frac{2S(n) - S(2n)}{9} \cdot 5 \implies S(2n) \geq \frac{S(n)}{5},$$

which is the claim. $\square$

Now check the three cases as suggested by Claim 1.

- if $S(2n) = 4$, then $S(n) = 164$;
- if $S(2n) = 10$, then $S(n) = 122$;
- if $S(2n) = 16$, then $S(n) = 44$.

Only the last case obeys Claim 2, so $S(2n) = 16$, $S(n) = 44$. In addition, all n with nonzero digits satisfying these two conditions are valid.

Repeating the proof of Claim 2, the multiplication $n \times 2$ requires exactly 8 carry-overs, so it is abundantly clear n has between 8 and 12 digits, inclusive. What remains is careful casework:

- If n has 8 digits, by stars-and-bars there are $\binom{4+8-1}{8-1} = 330$ solutions.
- If n has 9 digits, there are $\binom{9}{1}\binom{11}{8} = 1485$ solutions.
- If n has 10 digits, there are $\binom{10}{2}\binom{11}{9} = 2475$ solutions.
- If n has 11 digits, there are $\binom{11}{3}\binom{11}{10} = 1815$ solutions.
- If n has 12 digits, there are $\binom{12}{4}\binom{11}{11} = 495$ solutions.

The grand total is $330 + 1485 + 2475 + 1815 + 495 = 6600$.

§4 Relay problems

§4.1 Relay 1

1. Let n be a two-digit integer, and let m be the result when we reverse the digits of n. If $n - m$ and $n + m$ are both perfect squares, find n.

2. Let $T =$ TNYWR. Arrange the numbers 0, 1, 2, …, T in a circle. What is the expected number of (unordered) pairs of adjacent numbers that sum to T?

3. Let $T =$ TNYWR. Let $ABCD$ be a parallelogram with area 2020 such that $AB/BC = T$. The bisectors of $\angle DAB$, $\angle ABC$, $\angle BCD$, $\angle CDA$ form a quadrilateral. Compute the area of this quadrilateral.

§4.2 Relay 2

1. Compute the number of ordered triples (p, q, r) of primes, each at most 30, such that

$$p + q + r = p^2 + 4.$$

2. Let $T =$ TNYWR. Let ABC be a triangle with incircle ω. Points E, F lie on $\overline{AB}$, $\overline{AC}$ such that $\overline{EF} \parallel \overline{BC}$ and $\overline{EF}$ is tangent to ω. If $EF = T$ and $BC = T + 1$, compute $AB + AC$.

3. Let $T =$ TNYWR. There is a positive integer k such that T is the remainder when $17^0 + 17^1 + 17^2 + \cdots + 17^k$ is divided by 1000. Compute the remainder when 17^k is divided by 1000.

§5 Relay statistics

§5.1 Leaderboard

Note: Ties were broken by order of submission.

Usernames	Total	Set I	Set II
AlanHung	6	3	3
kvedula2004	5	0	5
Williamgolly, GoodInMathEverytime, AopsUser101	5	0	5
bissue	3	3	0
sriraamster, GerpTheGreat, onezero	0	0	0
kred9, lrjr24, CT17	0	0	0
usernameyourself, v4913, Mathletesv	0	0	0
superagh	0	0	0
Awesome_360, heavytoothpaste, bibear	0	0	0

§5.2 Answer Key

R1-1. 65

R1-2. $\frac{66}{65}$

R1-3. $\frac{101}{429}$

R2-1. 13

R2-2. 378

R2-3. 297

§6 Relay solutions

§6.1 Problem R1.1, by Raymond Feng

Let n be a two-digit integer, and let m be the result when we reverse the digits of n. If $n-m$ and $n+m$ are both perfect squares, find n.

Let $n = 10a + b$, $m = 10b + a$. Then $n - m = 9(a - b)$ and $n + m = 11(a + b)$. Since $a + b \le 18$ and $a + b$ is a multiple of 11 (else $n + m$ would only have one factor of 11), we have $a + b = 11$. But $a - b$ is a square, and the only (a, b) that works is $(6, 5)$.

The answer is 65.

§6.2 Problem R1.2, by Andrew Wen

Let T = TNYWR. Arrange the numbers 0, 1, 2, $\dots$, T in a circle. What is the expected number of (unordered) pairs of adjacent numbers that sum to T?

For each $0 \le n \le T$, there is a probability $1/T$ that the next number in the circle is $T - n$. By linearity of expectation, the expected number of such n is $(T + 1)/T$. With $T = 65$, the answer is $66/65$.

§6.3 Problem R1.3, by Justin Lee

Let T = TNYWR. Let $ABCD$ be a parallelogram with area 2020 such that $AB/BC = T$. The bisectors of $\angle DAB$, $\angle ABC$, $\angle BCD$, $\angle CDA$ form a quadrilateral. Compute the area of this quadrilateral.

The inner quadrilateral is a rectangle. Say $AB = x$, $BC = y$; wlog $x < y$. Then the distance between the bisectors of $\angle A$, $\angle C$ is $(y - x)\sin\frac{A}{2}$ and the distance between the bisectors of $\angle B$, $\angle D$ is $(y - x)\cos\frac{A}{2}$. Hence the area of the inner quadrilateral is

$$(y-x)^2 \sin\tfrac{A}{2}\cos\tfrac{A}{2} = \frac{(y-x)^2}{2xy}\cdot(xy\sin A) = \frac{1010(T-1)^2}{T}.$$

With $T = 66/65$, the answer is $101/429$.

§6.4 Problem R2.1, by Rishabh Das

Compute the number of ordered triples (p, q, r) of primes, each at most 30, such that

$$p + q + r = p^2 + 4.$$

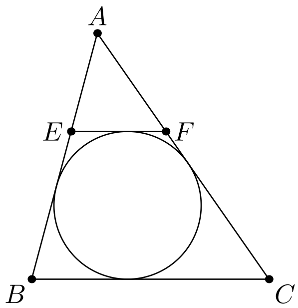
By $p^2 - p + 4 = q + r \le 60$, we have $p \le 7$.

- $p = 2$ gives $q + r = 6$, which has 1 solution $(q, r) = (3, 3)$.
- $p = 3$ gives $q + r = 10$, which has 3 solutions $(3, 7)$, $(5, 5)$, $(7, 3)$.
- $p = 5$ gives $q + r = 24$, which has 6 solutions $(5, 19)$, $(7, 17)$, $(11, 13)$, $(13, 11)$, $(17, 7)$, $(19, 5)$.
- $p = 7$ gives $q + r = 46$, which has 3 solutions $(17, 29)$, $(23, 23)$, $(29, 17)$.

The answer is 13.

§6.5 Problem R2.2, by Eric Shen

Let T = TNYWR. Let ABC be a triangle with incircle ω. Points E, F lie on $\overline{AB}$, $\overline{AC}$ such that $\overline{EF} \parallel \overline{BC}$ and $\overline{EF}$ is tangent to ω. If $EF = T$ and $BC = T+1$, compute $AB + AC$.



Since $\triangle AEF \sim \triangle ABC$, let $r := EF/BC = AE/AB = AF/AC$. Then $BE = (1-r)AB$, $CF = (1-r)AC$. Applying Pitot theorem on $BEFC$ gives

$$(1-r)(AB + AC) = (1+r)BC.$$

Setting $BC = T$, $r = (T-1)/T$, we have

$$AB + AC = (T+1) \cdot \frac{1 + \frac{T}{T+1}}{1 - \frac{T}{T+1}} = (T+1)(2T+1).$$

With $T = 13$, the answer is 378.

§6.6 Problem R2.3, by Allen Baranov

Let T = TNYWR. There is a positive integer k such that T is the remainder when $17^0 + 17^1 + 17^2 + \cdots + 17^k$ is divided by 1000. Compute the remainder when 17^k is divided by 1000.

By geometric series formula, we have

$$16T \equiv 17^{k+1} - 1 \pmod{1000} \implies 17^k \equiv T - \frac{T-1}{17} \pmod{1000}.$$

With $T = 378$, we have

$$\frac{T-1}{17} \equiv \frac{377}{17} \equiv \frac{377 \cdot 59}{3} \equiv 81 \pmod{1000},$$

so the answer is $378 - 81 \equiv 297$.

§7 Tiebreaker problems

1. Each of the six boxes shown in the equation below is replaced with a distinct number chosen from $\{1, 2, 3, \ldots, 27\}$.

$$S = \frac{\square}{\square} + \frac{\square}{\square} + \frac{\square}{\square}.$$

Suppose that the order of the fractions doesn't matter. Then there is exactly one way to arrange six numbers into the boxes such that $S < 1$ and S is as large as possible. Compute the sum of the 6 numbers.

2. The *AMC 12* consists of 25 problems, where for each problem, a correct answer is worth 6 points, leaving the problem blank is worth 1.5 points, and an incorrect answer is worth 0 points. The *AIME* consists of 15 problems, where each problem is worth 10 points, and no partial credit is given.

Any contestant who scores at least 84 on the AMC 12 is eligible for the AIME, and the USAMO index of such a student is the sum of his AMC 12 and AIME scores. Determine the smallest integer $N > 200$ such that no contestant can possibly obtain a USAMO index of $\frac{1}{2}N$.

2′. The *AMC 12* consists of 25 problems, where for each problem, a correct answer is worth 6 points, leaving the problem blank is worth 1.5 points, and an incorrect answer is worth 0 points. The *AIME* consists of 15 problems, where each problem is worth 10 points, and no partial credit is given.

Any contestant who scores at least 84 on the AMC 12 is eligible for the AIME, and the USAMO index of such a student is the sum of his AMC 12 and AIME scores. Determine the smallest integer $N > 300$ such that no contestant can possibly obtain a USAMO index of $\frac{1}{2}N$.

§8 Tiebreaker statistics

§8.1 Leaderboard

Username	Indiv	Time I	Time II	Time II′
kvedula2004	8	10:00	3:17	
—	8	10:00	6:00	
dchenmathcounts	7	10:00		3:52
MP8148	7	10:00		6:00
—	7	10:00		6:00

§8.2 Answer Key

TB-1. 104

TB-2. 202

TB-2′. 545

§9 Tiebreaker solutions

§9.1 Problem TB-1, by Kyle Lee

Each of the six boxes shown in the equation below is replaced with a distinct number chosen from $\{1, 2, 3, \ldots, 27\}$.

$$S = \frac{\square}{\square} + \frac{\square}{\square} + \frac{\square}{\square}.$$

Suppose that the order of the fractions doesn't matter. Then there is exactly one way to arrange six numbers into the boxes such that $S < 1$ and S is as large as possible. Compute the sum of the 6 numbers.

The answer is 104, attained by

$$S = \frac{12}{25} + \frac{1}{26} + \frac{13}{27} = \frac{25 \cdot 26 \cdot 27 - 1}{25 \cdot 26 \cdot 27}.$$

Since S is rational, the maximum possible denominator of S is $25 \cdot 26 \cdot 27$, thus the above value of S is maximal.

> **Remark.** In general,
> $$\frac{\frac{n-2}{2}}{n-1} + \frac{1}{n} + \frac{\frac{n}{2}}{n+1} = \frac{(n-1)n(n+1) - 1}{(n-1)n(n+1)}$$
> for even n.

§9.2 Problem TB-2, by Raymond Feng

The *AMC 12* consists of 25 problems, where for each problem, a correct answer is worth 6 points, leaving the problem blank is worth 1.5 points, and an incorrect answer is worth 0 points. The *AIME* consists of 15 problems, where each problem is worth 10 points, and no partial credit is given.

Any contestant who scores at least 84 on the AMC 12 is eligible for the AIME, and the USAMO index of such a student is the sum of his AMC 12 and AIME scores. Determine the smallest integer $N > 200$ such that no contestant can possibly obtain a USAMO index of $\frac{1}{2}N$.

The answer is 202.

- 201 works by having an AMC score of 100.5 (say, 16 correct and 3 blank) and an AIME score of 0.
- 202 fails, since the corresponding AMC score must be an element of $\{91, 101\}$, neither of which is divisible by 1.5.

§9.3 Problem TB-2′, by Raymond Feng

The *AMC 12* consists of 25 problems, where for each problem, a correct answer is worth 6 points, leaving the problem blank is worth 1.5 points, and an incorrect answer is worth 0 points. The *AIME* consists of 15 problems, where each problem is worth 10 points, and no partial credit is given.

Any contestant who scores at least 84 on the AMC 12 is eligible for the AIME, and the USAMO index of such a student is the sum of his AMC 12 and AIME scores. Determine the smallest integer $N > 300$ such that no contestant can possibly obtain a USAMO index of $\frac{1}{2}N$.

The answer is 545. Let $S \geq 84$ be the contestant's AMC 12 score and let T be his AIME score. Evidently $2S$ must be an integer multiple of 3.

Claim. The minimum multiple of 1.5 that S cannot attain is 142.5.

Proof. Let c be the number of correct answers and b the number of blank answers, so $S = 6c + 1.5b$. First we check that all $S \leq 141$ are achievable:

- If $c \leq 22$, then $b \in \{0, 1, 2, 3\}$ are allowed, so $S \in \{6c, 6c + 1.5, 6c + 3, 6c + 4.5\}$ are all valid; then S can achieve all multiples of 1.5 up to 136.5.
- If $c = 23$, then $b \in \{0, 1, 2\}$, so $S \in \{138, 139.5, 141\}$ are valid.

Finally, we show 142.5 is not achievable:

- If $c \geq 24$, then $S \geq 144$.
- If $c = 23$, then $S = 142.5$ implies $b = 3$, which is impossible.
- If $c \leq 22$, then $S \leq 6c + 1.5(25 - c) = 4.5c + 37.5 \leq 136.5$.

The claim is proven. □

For $233 \leq \frac{1}{2}N \leq 271$, one of $\frac{1}{2}N - 130$, $\frac{1}{2}N - 140$, $\frac{1}{2}N - 150$ is a multiple of 1.5 in $[84, 141.5]$, so such $\frac{1}{2}N$ are attainable. It is analogous to check N with $150 \leq \frac{1}{2}N < 233$ also work.

Otherwise, say $\frac{1}{2}N = 272.5$; then $T \geq \frac{1}{2}N - 150 = 122.5$, so $T \in \{130, 140, 150\}$. The possible values of S are $\{122.5, 132.5, 142.5\}$, neither of which works.

Remark. A value of S is valid if and only if

- $S \notin \{142.5, 147, 148.5\}$, and
- $2S$ is an integer multiple of 3.