2020 CAMO/CJMO
Recommended Marking Scheme
January 10 to February 3, 2020

Test schema

- 1st Christmas American Math Olympiad: BDE HIJ
- 3rd Christmas Junior Math Olympiad: ABC FGI

Marking scheme, Day 1

A. Let N be a positive integer, and let S be the set of all tuples with positive integer elements and a sum of N. For instance, $t_1 = (N)$, $t_2 = (1, 1, N-2)$, $t_3 = (1, N-1)$, and $t_4 = (N-1, 1)$ are all distinct tuples in S. For all tuples t, let $p(t)$ denote the product of all the elements of t. For instance, $p(t_1) = N$, $p(t_2) = N-2$, and $p(t_3) = p(t_4) = N-1$.

Evaluate the expression (where we sum over all elements t of S)

$$\sum_{t \in S} p(t).$$

First solution. Let $F_0 = 0$, $F_1 = 1$, and for all $k \geq 2$, $F_k = F_{k-1} + F_{k-2}$. The answer is F_{2N}.

1 point for claiming the correct answer.

To show this, we use strong induction. The base case, $N = 1$, is clear. Let $f(N)$ be the answer for N. It can be seen that if the hypothesis holds for all integers less than k, then by picking the first element of the tuple first, $f(k)$ is equal to

$$f(k) = \sum_{i=0}^{k-1} (k-i) F_{2i} = \sum_{i=0}^{k-1} \sum_{j=0}^{i} F_{2j} = \sum_{i=0}^{k-1} F_{2i+1} = F_{2k},$$

and the induction is complete.

6 points for the inductive step. Deduct 1 point if the base case is not acknowledged, and award 1 point if the student proves

$$f(k) = \sum_{i=0}^{k-1} \sum_{j=0}^{i} F_{2i+1}$$

without further progress.

Second solution. Let $F_0 = 0$, $F_1 = 1$, and for all $k \geq 2$, $F_k = F_{k-1} + F_{k-2}$. The answer is F_{2N}.

For all positive integers i, let S_i denote the subset of S that contains all tuples with cardinality i.

Claim. For all i,

$$\sum_{t\in S_i} p(t) = \binom{N-1+i}{2i-1}.$$

1 point for establishing this claim.

First proof by combinatorial argument. The desired sum is bijective with splitting up a line of N items into i sections, and picking a representative from each section. Using the Stars and Bars method, we can add in $i-1$ dividers. We can pick $2i-1$ items, each of which is either a representative or divider. Since between two representatives there is exactly one divider, which of these selected items is a divider follows. Hence, there are $\binom{N-1+i}{2i-1}$ ways to pick sections and representatives, as desired. □

4 points for a complete combinatorial argument.

Second proof by strong induction. Let $S_i(k)$ be the number of tuples with cardinality i whose elements sum to k. It suffices to show that

$$\sum_{t\in S_i(k)} p(t) = \binom{k-1+i}{2i-1}.$$

The base case, $i=1$, is trivial. Then, we can pick the first element of each tuple first, so by the Hockey Stick Identity,

$$\begin{aligned}\sum_{t\in S_i(k+1)} p(t) &= \sum_{j=1}^{k-i+2}\left(j\sum_{t\in S_i(k+1-j)} p(t)\right) = \sum_{j=1}^{k-i+2}\left(j\binom{k+i-j}{2i-1}\right)\\ &= \sum_{\ell=2i-1}^{k+i-1}\sum_{j=2i-1}^{\ell}\binom{j}{2i-1} = \sum_{\ell=2i-1}^{k+i-1}\binom{\ell+1}{2i} = \binom{k+1+i}{2i+1},\end{aligned}$$

as required.

4 points for rigorous and clear induction steps. Award 1 point if the student proves

$$\sum_{t\in S_i(k+1)} p(t) = \sum_{\ell=2i-1}^{k+i-1}\sum_{j=2i-1}^{\ell}\binom{j}{2i-1}$$

without further progress.

□

We then have that

$$\sum_{t\in S} p(t) = \sum_{i=1}^{N}\sum_{t\in S_i} p(t) = \sum_{i=1}^{N}\binom{N-1+i}{2i-1} = \sum_{i=1}^{N}\binom{N-1+i}{N-i} = F_{2N},$$

as desired.

2 points for completing the proof. The Fibonacci identity

$$\sum_{i=1}^{n} \binom{N-1+i}{N-1} = F_{2N}$$

may be cited without proof.

B. Let $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ (meaning f takes positive real numbers to positive real numbers) be a nonconstant function such that for any positive real numbers x and y,

$$f(x)f(y)f(x+y) = f(x) + f(y) - f(x+y).$$

Prove that there is a constant $a > 1$ such that

$$f(x) = \frac{a^x - 1}{a^x + 1}$$

for all positive real numbers x.

Solution. Rewrite our functional equation as

$$f(x+y) = \frac{f(x) + f(y)}{1 + f(x)f(y)}.$$

The key claim is that $f(x) < 1$ or $f \equiv 1$.

Claim. $f(x) \geq 1 \implies f(\frac{x}{2}) = 1$.

Proof. Plugging in $(\frac{x}{2}, \frac{x}{2})$ into our functional equation gives

$$\frac{2f(\frac{x}{2})}{1 + f(\frac{x}{2})^2} = f(x) \geq 1 \implies \left(f\left(\frac{x}{2}\right) - 1\right)^2 \leq 0 \implies f\left(\frac{x}{2}\right) = 1,$$

as desired. □

1 point for this claim.

Claim. $f(x) \geq 1 \implies f \equiv 1$.

Proof. By Claim 1, there exists y such that $f(y) = 1$. Furthermore, if $f(y) = 1$ then $f(\frac{y}{2}) = 1$ by Claim 1, so we can take y infinitely small. Then, by our functional equation,

$$f(x+y) = \frac{f(x) + 1}{1 + f(x)} = 1$$

for all x, so $f \equiv 1$. □

1 point for this claim. If the conclusion $f(x) \geq 1 \implies f \equiv 1$ is proven without the first claim, award 2 points in total.

Now, discard the trivial solution $f \equiv 1$. We have that $f(x) < 1$ for all x. Let

$$g(x) = \ln\left(\frac{1 + f(x)}{1 - f(x)}\right).$$

Then,

$$g(x+y) = \ln\left(\frac{(1 + f(x))(1 + f(y))}{(1 - f(x))(1 - f(y))}\right) = g(x) + g(y),$$

3 points for successfully defining either $g(x) = \operatorname{arctanh}(f(x))$ or

$$g(x) = \ln\left(\frac{1 + f(x)}{1 - f(x)}\right),$$

and showing g obeys Cauchy's Functional Equation.

so g satisfies Cauchy's Functional Equation. Since $f(x) > 0$, $g(x) > 0$, so g is bounded and there exists a positive constant k such that $g(x) = kx$. Thus $a = e^k$, and we are done.

2 points for concluding that g must be linear and finishing the proof.

C. Let ABC be an acute triangle with circumcenter O, orthocenter H, and $\angle A = 45^\circ$. Denote by M the midpoint of $\overline{BC}$, and let P be a point such that $\overline{AP}$ is parallel to $\overline{BC}$ and $\angle HMB = \angle PMC$. Show that if segment OP intersects the circle with diameter $\overline{AH}$ at Q, then $\overline{OA}$ is tangent to the circumcircle of $\triangle APQ$.

Solution. Define ω as the circumcircle of ABC, and let A' be its second intersection with $\overline{AP}$, H' be its second intersection with $\overline{AH}$, and X be its second intersection with $\overline{H'M}$. Furthermore, define Q' as the second intersection of the circumcircles of $\triangle H'HO$ and $\triangle A'AO$.

1 point for the construction of $(H'HO)$ and $(A'AO)$; award these points if they appear on the provided diagram. (The motivation for these circles are to try to show P and Q are inverses with respect to the circumcircle.)

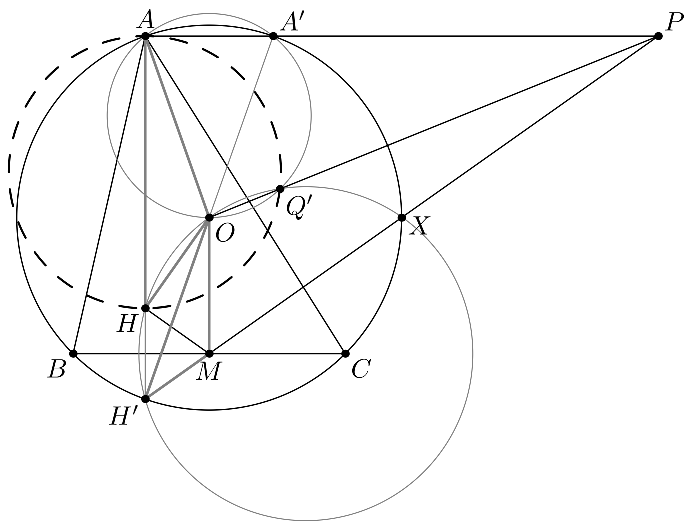
Claim. Q' lies on the circle with diameter $\overline{AH}$.

Proof. First observe that H' is the antipode of A', so

$$\angle HQO = \angle HH'O = 90^\circ - \angle AA'O = 90^\circ - \angle AQ'O$$

and $\angle AQ'H = 90^\circ$. This immediately proves the assertion. ☐

1 point for using phantom points to show $Q = (H'HO) \cap (A'AO)$ lies on (AH).



Claim. X lies on the circumcircle of $H'HO$.

Proof. Reflect O over $\overline{BC}$ to O'. Since $\angle BOC = 2\angle BAC = 90^\circ$, O lies on the circle with diameter $\overline{BC}$. Note that H and H' are reflections across $\overline{BC}$, so the circumcenter of $\triangle H'HO$ lies on $\overline{BC}$. This immediately yields that O' also lies on $(H'HO)$, and thus

$$MH' \cdot MX = MB \cdot MC = MO \cdot MO',$$

and the desired conclusion follows readily. $\square$

3 points for using $\angle A = 45^\circ$ to show X lies on $(H'HO)$.

Finally, because H' is the reflection of H over $\overline{BC}$, H' must lie on $\overline{PM}$ as well; hence, P is the radical center of $(A'AO)$, (ABC), and $(H'HO)$. It follows that Q' lies on $\overline{OP}$, and therefore $Q' = Q$. Noting that $\angle OQA = \angle OA'A = \angle OAP$ solves the problem.

1 point for showing P is a radical center, and 1 point for concluding the proof (i.e. showing $Q' = Q$.)

D. Let k be a positive integer, $p > 3$ a prime, and n an integer with $0 \le n \le p^{k-1}$. Prove that

$$\binom{p^k}{pn} \equiv \binom{p^{k-1}}{n} \pmod{p^{2k+1}}.$$

Solution. We use falling factorial notation:

$$(x)_n = x(x-1)(x-2)\cdots(x-(n-1)).$$

First, a lemma:

Lemma (Falling factorial congruence)**.** *For $p > 3$ and $i < n$, we have*

$$\left(p^k - pi - 1\right)_{p-1} \equiv \left(p(i+1) - 1\right)_{p-1} \pmod{p^{k+2}}.$$

1 point for stating the lemma without proof.

Proof. Expand the left-hand side and remove all multiples of p^{k+2} to obtain

$$\left(p(i+1)-1\right)_{p-1} + p^k \left(p(i+1)-1\right)_{p-1} \left[\sum_{j=1}^{p-1} \frac{1}{pi+j}\right] \pmod{p^{k+2}},$$

so it suffices to verify the bracketed term is $0 \pmod{p^2}$.

2 points for reducing the lemma to a generalized version of Wolstenholme's theorem:

$$\sum_{j=1}^{p-1} \frac{1}{pi+j} \equiv 0 \pmod{p^2}.$$

The bracketed term equals

$$\begin{aligned}\sum_{j=1}^{p-1} \frac{1}{pi+j} &\equiv \sum_{j=1}^{\frac{p-1}{2}} \left(\frac{1}{pi+j} + \frac{1}{p(i+1)-j}\right) \\ &\equiv p(2i+1)\left[\sum_{j=1}^{\frac{p-1}{2}} \frac{1}{(pi+j)\big(p(i+1)-j\big)}\right] \pmod{p^2},\end{aligned}$$

so we need the new bracked term to be $0 \pmod{p}$. It equals

$$\sum_{j=1}^{\frac{p-1}{2}} \frac{1}{(pi+j)\big(p(i+1)-j\big)} \equiv -\sum_{j=1}^{\frac{p-1}{2}} \frac{1}{j^2} \equiv -\frac{1}{2}\sum_{j=1}^{p-1} \frac{1}{j^2} \equiv -\frac{1}{2}\sum_{j=1}^{p-1} j^2 \equiv 0 \pmod{p}$$

when $p > 3$, as desired. ☐

2 points for proving the generalized Wolstenholme's theorem and thus the lemma.

Rewrite the desired as

$$\begin{aligned}\frac{\left(p^k\right)_{pn}}{(pn)!} &\equiv \frac{\left(p^{k-1}\right)_n}{n!} \pmod{p^{2k+1}} \\ \iff \left(p^k\right)_{pn} \cdot n! &\equiv \left(p^{k-1}\right)_n (pn)! \pmod{p^{2k+n+1}} \\ \Longleftarrow \left(p^k-1\right)_{pn-1} \cdot (n-1)! &\equiv \left(p^{k-1}-1\right)_{n-1} (pn-1)! \pmod{p^{k+n+1}}.\end{aligned}$$

1 point for rearranging the given expression to something more workable.

With some rearranging, the left-hand sign becomes

$$p^{n-1}(n-1)!\left(p^{k-1}-1\right)_{n-1} \prod_{i=0}^{n-1} \left(p^k - pi - 1\right)_{p-1} \pmod{p^{k+n+1}},$$

and the right-hand sign becomes

$$p^{n-1}(n-1)!\left(p^{k-1}-1\right)_{n-1} \prod_{i=0}^{n-1} \left(p(i+1)-1\right)_{n-1} \pmod{p^{k+n+1}}.$$

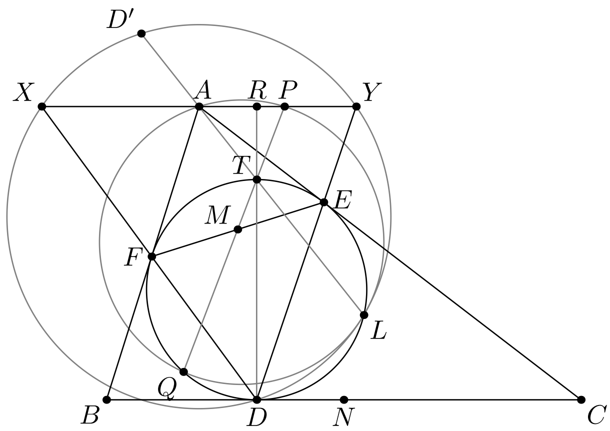
Since both expressions already carry a p^{n-1} term, they are equal by the lemma.

1 point for reducing the problem to proving

$$\prod_{i=0}^{n-i} \left(p^k - pi - 1\right)_{p-1} \equiv \prod_{i=1}^{n-1} \left(p(i+1)-1\right)_{n-1} \pmod{p^{k+2}}.$$

E. Let ABC be a triangle with incircle ω, and let ω touch $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Point M is the midpoint of $\overline{EF}$, and T is the point on ω such that $\overline{DT}$ is a diameter. Line MT meets the line through A parallel to $\overline{BC}$ at P and ω again at Q. Lines DF and DE intersect line AP at X and Y respectively. Prove that the circumcircles of $\triangle APQ$ and $\triangle DXY$ are tangent.

Solution.



Let $\overline{DT}$ intersect $\overline{AP}$ at R, and let $\overline{AT}$ intersect ω again at L.

Claim. T lies on $\overline{EX}$ and $\overline{FY}$, T is the orthocenter of $\triangle DXY$, and A is the midpoint of $\overline{XY}$.

Proof. Redefine $X = \overline{DF} \cap \overline{TE}$ and $Y = \overline{DE} \cap \overline{TF}$. Since $\angle DET = \angle DFT = 90°$, T is the orthocenter of $\triangle DXY$. Thus, $\overline{DT} \perp \overline{XY}$, so $\overline{XY} \parallel \overline{BC}$.

By the Three Tangents lemma, the tangents to ω at E and F intersect at the midpoint of $\overline{XY}$; but this is A, thus recovering the original definitions of X and Y. □

1 points for proving this claim.

Claim. L lies on (DXY) and (APQ).

Proof. Since A is the midpoint of $\overline{XY}$ and T is the orthocenter of $\triangle DXY$, $\overline{AT}$ passes through D', the antipode of D on (DXY). Note that $\angle DLD' = \angle DLT = 90°$, so L lies on (DXY).

Now $L \in (APQ)$ follows from $TA \cdot TL = TR \cdot TD = TP \cdot TQ$, thus proving the claim. □

2 points for proving L is the common point of (DXY) and (APQ).

Finally, since $\overline{TM}$ is the T-symmedian of $\triangle TXY$ and L is the Miquel point of $XYEF$,

$$\frac{LX}{LY} = \frac{XF}{YE} = \frac{TX}{TY} = \left(\frac{PX}{PY}\right)^2.$$

It follows that $\overline{LP}$ is a symmedian of $\triangle LXY$. Since $\overline{LA}$ and $\overline{LP}$ are isogonal, (LAP) and (DXY) are tangent, and we are done.

4 points are possible here: 3 points for proving $\overline{LP}$ is a symmedian of $\triangle LXY$, and 1 point for concluding that (DXY) and (APQ) are tangent at L.

Marking scheme, Day 2

F. For all positive integers k, define $s(k)$ to be the result when the last digit of k is moved to the front of k. For instance, $s(2020) = 202$ and $s(1234) = 4123$. For each positive integer n, find the number of positive integers $k < 10^n$ that satisfy $s(9k) = 9s(k)$.

Solution. First observe that all multiples of 10 work: if $k = 10t$, then it is easy to see that $s(9k) = 9t = 9s(k)$. Say a positive integer k is *n-good* if it not divisible by 10, has n digits, and satisfies $s(9k) = 9s(k)$. The key is to characterize all n-good integers.

Award no points for noting that multiples of 10 work.

Claim (Characterizing good integers)**.** A positive integer k is n-good if and only if

- $10^{n-1} \le k < 10^n/9$, and
- the units digit of k is 1.

1 point for correctly describing (without proof) all good integers.

Proof. The two bullet-points are equivalent to

$$k = 10^{n-1} + 10\ell + 1$$

for some $\ell < 10^{n-2}/9$. If k is of this form, then ℓ and 9ℓ have the same number of digits. Thus

$$s(9k) = 9 \cdot 10^{n-1} + 9 \cdot 10^{n-2} + 9\ell = 9s(k),$$

and k is good.

1 point for proving all numbers of this form are good

Conversely, assume that k is n-good. Let the units digit of k be d. There are two cases to consider.

- Suppose $d = 1$. Then $9k$ has a units digit of 9, so $9s(k) = s(9k)$ has a leading digit of 9. It follows that $9s(k) = s(9k)$ also has n-digits, so $9k$ has n digits, and $k < 10^n/9$.
- Suppose $d \neq 1$. Since $9k \equiv -k \pmod{10}$, the units digit of $9k$ is $10 - d$, so the leading digit of $s(9k)$ is also $10 - d$.

  The leading digit of $s(k)$ is $d \neq 1$, so the leading digit of $9s(k)$ is either d or $d - 1$. Note that $d - 1 = 10 - d$ is impossible, so $d = 10 - d$, and $d = 5$.

  Let e be the tens digit of k. This becomes the units digit of $s(k)$, so the units digit of $9s(k)$ is $10 - e$. However the units digit of $s(9k)$ is the tens digit of $9k$, which is $4 + (10 - e) \pmod{10}$, contradiction. Thus this case is impossible.

Combining the above arguments, the claim has been proven. □

3 points for proving the converse.

The number of multiples of 10 less than 10^n is $10^{n-1} - 1$. Let $f(m)$ be the number of m-good integers for $m \ge 2$. Note that if all m digits are 1, then the integer is good. Otherwise there must be a 0. Suppose there are k digits after the 0. All the digits before the 0 must be 1, and we have 10^{k-1} choices for the digits after the 0 but before the units digit.

Therefore we may compute

$$f(m) = 1 + \sum_{i=0}^{m-3} 10^{i-1} = \frac{10^{m-2}+8}{9}.$$

1 point for correctly computing $f(m)$.

Clearly $f(1) = 1 = \frac{1+8}{9}$. Finally

$$\sum_{m=1}^{n} f(m) = \frac{8n}{9} + \frac{1}{9}\left(1 + \sum_{i=0}^{n-2} 10^i\right) = \frac{10^{n-1}+72n+8}{81},$$

and after adding back multiples of 10, the answer is

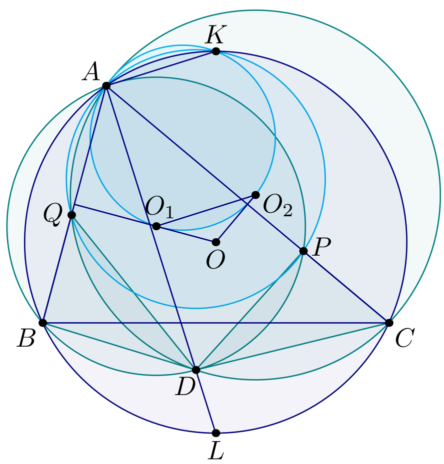
$$10^{n-1} - 1 + \frac{10^{n-1}+72n+8}{81} = \boxed{\frac{82\cdot 10^{n-1}+72n-73}{81}},$$

and we are done.

1 point for the correct answer. In total, 2 points were awarded for (i) correctly computing $f(m)$, and (ii) the correct anwer. If the student does not realize multiples of 10 work, award 1 of these 2 points.

G. Let ABC be a triangle, and D be a point on the internal angle bisector of $\angle BAC$ but not on the circumcircle of $\triangle ABC$. Suppose that the circumcircle of $\triangle ABD$ intersects $\overline{AC}$ again at P and the circumcircle of $\triangle ACD$ intersects $\overline{AB}$ again at Q. Denote by O_1 and O_2 the circumcenters of $\triangle ABD$ and $\triangle ACD$, respectively. Prove that the circumcenters of $\triangle ABC$, $\triangle APQ$, and $\triangle AO_1O_2$ are collinear.

Solution.



Let K be the midpoint of arc BAC on (ABC). I claim that (APQ) and (AO_1O_2) pass through K, from which the result is obvious.

1 point for claiming the midpoint of arc BAC is the common point of (APQ) and (AO_1O_2).

Claim. K lies on (APQ).

Proof. By construction, D is the center of spiral similarity sending $\overline{BQ}$ to $\overline{PC}$. However, since $\overline{AD}$ bisects $\angle BAP$, $DB = DP$, so $\triangle DBQ \cong \triangle DPC$, and $BQ = PC$.

1 point for proving $BQ = PC$.

Since $\measuredangle QBK = \measuredangle ABK = \measuredangle ACK = \measuredangle PCK$, by SAS, $\triangle KBQ \cong \triangle KCP$, so K is the Miquel point of $BCPQ$, and K lies on (APQ), as desired. $\square$

2 points for proving K lies on (APQ).

Claim. $OO_1 = OO_2$, where O is the circumcenter of $\triangle ABC$.

Proof. Note that $\overline{O_1O_2}, \overline{OO_1}, \overline{OO_2}$ are the perpendicular bisectors of $\overline{AD}, \overline{AB}, \overline{AC}$, respectively, so

$$\measuredangle(\overline{OO_1}, \overline{O_1O_2}) = \measuredangle(\overline{AB}, \overline{AD}) = \measuredangle(\overline{AD}, \overline{AC}) = \measuredangle(\overline{O_1O_2}, \overline{OO_2}),$$

as required. $\square$

1 point for proving $OO_1 = OO_2$.

Claim. K lies on (AO_1O_2).

Proof. Since $\overline{O_1O_2}$ and $\overline{AK}$ are both perpendicular to $\overline{AD}$, and O lies on both of their perpendicular bisectors, AO_1O_2K must be an isosceles trapezoid, so it is cyclic. $\square$

1 point for proving K lies on (AO_1O_2).

Hence, (ABC), (APQ), and (AO_1O_2) are coaxial, so their centers are collinear, as desired.

1 point for the conclusion that coaxial circles have collinear centers.

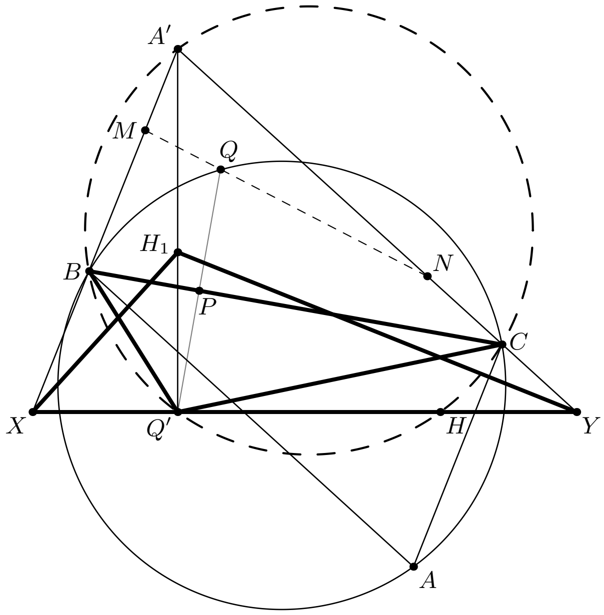
H. Let ABC be a triangle and Q a point on its circumcircle. Let E and F be the reflections of Q over $\overline{AB}$ and $\overline{AC}$, respectively. Select points X and Y on line EF such that $\overline{BX} \parallel \overline{AC}$ and $\overline{CY} \parallel \overline{AB}$, and let M and N be the reflections of X and Y over B and C respectively. Prove that M, Q, N are collinear.

First solution, by spiral similarity. Let A' be the point such that $ABA'C$ is a parallelogram, so $X \in \overline{A'B}$ and $Y \in \overline{A'C}$. Define H as the orthocenter of $\triangle ABC$, H_1 as the orthocenter of $\triangle A'XY$, Q' as the reflection of Q over $\overline{BC}$, and P as the foot of Q on $\overline{BC}$.

To begin, observe that Q' must lie on $\overline{XY}$ by homothety on a Simson line. This, in conjunction with $\angle BQC = \angle BHC$ and the well-known fact that H lies on $\overline{XY}$ implies that Q' lies on $(A'BHC)$, so it must be the foot of A' on $\overline{XY}$.

1 point for showing H lies on $\overline{XY}$.

1 point for proving Q' is the foot of A' on $\overline{XY}$.



Next, we have

$$\measuredangle H_1XY = 90^\circ - \measuredangle XYA = \measuredangle CA'Q' = -\measuredangle Q'BC,$$

and similarly $\measuredangle XYH_1 = -\measuredangle BCQ'$; therefore, $\triangle Q'BC \sim \triangle H_1XY$ and, as a consequence, degenerate triangles PBC and $Q'XY$ are also similar.

3 points for proving $\triangle Q'BC \sim \triangle H_1XY$ and 1 point for proving $\triangle PBC \sim \triangle Q'XY$.

Collinearity of M, Q, N follows from the mean geometry theorem.

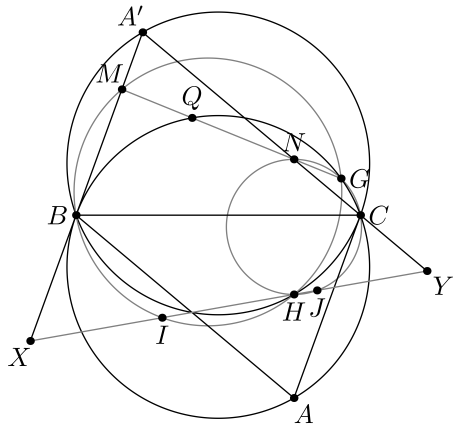
1 point for concluding by mean geometry theorem / spiral similarity.

Second solution, by angle chasing. Let A' be the point such that $ABA'C$ is a parallelogram, so $X \in \overline{A'B}$ and $Y \in \overline{A'C}$, and let O and H be the circumcenter and orthocenter of $\triangle ABC$. Since $\overline{XY}$ is the image of the Simson line from Q under homothety $(Q, 2)$[1], we know H lies on $\overline{XY}$.

1 point for showing H lies on $\overline{XY}$.

Let I and J be the projections of M and N onto $\overline{XY}$, so that B is the center of (MXI) and C is the center of (NYJ). Then $HIBM$ and $HJCN$ are cyclic with diameters $\overline{HM}$ and $\overline{HN}$; say they intersect again at G.

[1] sometimes called the *Steiner line*



1 point for constructing I and J. This illustrates that the student has exercised a soft skill: how can we use the conditions $BM = BX$ and $CN = CY$?

Then $\angle MGH = \angle NGH = 90^\circ$, so $G \in \overline{MN}$. Furthermore

$$\measuredangle BGM = \measuredangle IGB = \measuredangle IHB \quad \text{and} \quad \measuredangle NGC = \measuredangle CHJ.$$

Adding these, $\measuredangle BGC = \measuredangle CHB = \measuredangle BAC$, so G lies on (ABC).

3 points for showing G lies on (ABC).

Say $\overline{MN}$ intersects (ABC) again at Q' and $\overline{XY}$ intersects $(A'BC)$ again at D. Then recalling that $\measuredangle BGM = \measuredangle IHB$,

$$\measuredangle BCQ' = \measuredangle BGQ' = \measuredangle BGM = \measuredangle IHB = \measuredangle DHB = \measuredangle DCB,$$

and similarly $\measuredangle Q'BC = \measuredangle CBD$, so Q' and D are reflections across $\overline{BC}$, and $\overline{XY}$ is the Steiner line of Q'.

2 points for showing Q' and D are reflections across $\overline{BC}$, thus concluding the proof.

Third solution, by length. Let H be the orthocenter of $\triangle ABC$ and D the reflection of Q over $\overline{BC}$. Since $\overline{XY}$ is the image of the Simson line from Q under homothety $(Q, 2)$, we know H and D lie on $\overline{XY}$.

Let U and V lie on $\overline{XY}$ such that $\overline{BU}$ and $\overline{CV}$ are perpendicular to $\overline{BC}$.

Claim. $DU : DV = DX : DY$.

Proof. Let D' be the foot from Q to $\overline{BC}$ (i.e. the midpoint of $\overline{QD}$). Remark that $\overline{A'H}$ is a diameter of $(A'BC)$ by orthocenter reflections, so D is the foot from A' to $\overline{XY}$. Note that

$$\frac{DU}{DV} = \frac{D'B}{D'C} = \frac{DB}{DC} \cdot \frac{\cos \angle DBC}{\cos \angle DCB} = \frac{DB}{DC} \cdot \frac{\cos \angle DA'C}{\cos \angle DA'B} = \frac{DB}{DC} \cdot \frac{\sin \angle A'YD}{\sin \angle A'XD},$$

but

$$\frac{DX}{DY} = \frac{A'X}{A'Y} \cdot \frac{\sin \angle BA'D}{\sin \angle CA'D} = \frac{A'X}{A'Y} \cdot \frac{DB}{DC} = \frac{DB}{DC} \cdot \frac{\sin \angle A'YD}{\sin \angle A'XD},$$

as claimed. $\square$

Let X' and Y' be the reflections of X and Y over $\overline{BC}$. We have

$$\frac{MX'}{NY'} = \frac{UX}{VY} = \frac{DX}{DY} = \frac{QX'}{QY'},$$

so $\overline{X'Y'} \cap \overline{MN}$ is the reflection of D across $\overline{BC}$, which is Q. This completes the proof.

7 points if the length chase is successful; 0 points otherwise.

I. Let $f(x) = x^2 - 2$. Prove that for all positive integers n, the polynomial

$$P(x) = \underbrace{f(f(\ldots f}_{n \text{ times}}(x)\ldots)) - x$$

can be factored into two polynomials with integer coefficients and equal degree.

Solution. We first prove a lemma.

Lemma. *Let P be a monic polynomial. If P^2 has integer coefficients, then so does P.*

Proof. Suppose there is a polynomial without this property, and henceforth let P be such a polynomial of minimal degree. Note $P^2 \in \mathbb{Z}[x]$, so let the factorization of P^2 into factors that are powers of irreducible polynomials in $\mathbb{Z}[x]$ be $P_1, P_2, \ldots, P_k$.

By construction they do not share roots with one another. Since they multiply to the square of a polynomial in $\mathbb{R}[x]$, they are all squares in $\mathbb{R}[x]$.

1 point for proving P_i must all be squares in $\mathbb{R}[x]$.

If $k > 1$ they must all be in $\mathbb{Z}[x]$ by the minimality of the degree of P.

1 point for reducing to $k = 1$.

Hence $k = 1$ and $P(x)^2 = Q(x)^r$ for some r. If r is even we are done, so assume r is odd. Then $Q(x)$ must be the square of a polynomial in $\mathbb{R}[x]$, but it is irreducible, contradiction. □

2 points for reaching the desired contradiction.

Consider the sequence defined by $y_n = \frac{1}{2} f^n(2x)$. For $n > 0$,

$$y_n = \tfrac{1}{2}(f^{n-1}(2x)^2 - 2) = \tfrac{1}{2}(4y_{n-1}^2 - 2) = 2y_{n-1}^2 - 1.$$

If $|y_0| < 1$, say that $y_0 = \cos\theta$ for some angle θ. It follows that $y_n = \cos(2^n\theta)$ for all n, whence solutions to $P(x) = 0$ obey $\cos(2^n\theta) = \cos\theta$. Thus the set of solutions to $P(x) = 0$ includes

$$2\cos\left(\frac{2\pi k}{2^n - 1}\right) \quad \text{and} \quad 2\cos\left(\frac{2\pi k}{2^n + 1}\right) \quad \text{for all } k.$$

The former describes 2^{n-1} distinct roots and the latter describes $2^{n-1} + 1$ distinct roots. The only root they share is 1, so we have described all 2^n solutions.

1 point for describing all roots of f.

Claim. Let m be an odd integer. The monic polynomial with roots $2\cos(\frac{2\pi k}{m})$, $0 \le k < m$, has integer coefficients.

Proof. Let $g_n(2\cos\theta) = 2\cos(n\theta)$. The key observation is that

$$S_n(x) = xS_{n-1}(x) - S_{n-2}(x).$$

Indeed, this rewrites to

$$2\cos\theta\cos(n-1)\theta = \cos n\theta + \cos(n-2)\theta,$$

which is just the product-to-sum identity. With this, g_n is a monic integer polynomial of degree n for all n, but the polynomial $g_m(x)$ is exactly the polynomial we need. □

1 point for proving the claim.

The polynomial described by the above claim is precisely the square of the polynomial with roots $2\cos(\frac{2\pi k}{m})$, $0 \le k < \frac{m}{2}$, whence it has integer coefficients by the lemma. Let Q be this integer polynomial for $m = 2^n + 1$ and R for $m = 2^n - 1$.

Clearly P is monic. We can factor out $x-2$ from Q and add it to R (thus 2 is a double root), thereby giving two factors of P with integer coefficients and equal degree.

1 point for finishing the proof.

J. Let n be a positive integer. Eric and a squid play a turn-based game on an infinite grid of unit squares. Eric's goal is to capture the squid by moving onto the same square as it.

Initially, all the squares are colored white. The squid begins on an arbitrary square in the grid, and Eric chooses a different square to start on. On the squid's turn, it performs the following action exactly 2020 times: it chooses an adjacent unit square that is white, moves onto it, and sprays the previous unit square either black or gray. Once the squid has performed this action 2020 times, all squares colored gray are automatically colored white again, and the squid's turn ends. If the squid is ever unable to move, then Eric automatically wins. Moreover, the squid is claustrophobic, so at no point in time is it ever surrounded by a closed loop of black or gray squares. On Eric's turn, he performs the following action at most n times: he chooses an adjacent unit square that is white and moves onto it. Note that the squid can trap Eric in a closed region, so that Eric can never win.

Eric wins if he ever occupies the same square as the squid. Suppose the squid has the first turn, and both Eric and the squid play optimally. Both Eric and the squid always know each other's location and the colors of all the squares. Find all positive integers n such that Eric can win in finitely many moves.

Solution. Let $s = 2020$. In general, the answer for $s \ge 8$ is $n \ge 2s - 5$. Henceforth, by "distance," we refer to the length of the shortest path between Eric and the squid that does not intersect the squid ink. For all shown diagrams, a white circle represents Eric's initial position, a black circle represents the squid's initial position, a gray line represents Eric's path, and a solid line represents the squid's path.

1 point for the correct answer.

Proof of upper bound. Say $n < 2s - 5$. The key here is that Eric cannot get close enough to the squid, or the squid can surround Eric. We use the following estimate.

Claim. Suppose it's the squid's turn, the squid has only used gray ink (so there are no black squares), and the distance between Eric and the squid is $d \le s - 6$. Then the squid wins.

Proof. Consider the following picture.

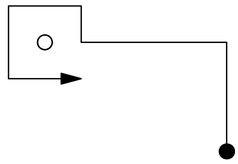


Figure 1: Surrounding Eric

Here it takes the squid $d - 1$ moves to get to the closest point adjacent to Eric, and then 7 moves to surround him. Hence if $s \ge d + 6$, the squid can surround Eric using black ink, as claimed. $\square$

Now assume that at an arbitrary point in time, Eric is a distance of $d \ge s - 5$ away, and it's the squid's turn. As the squid continues moving, it is able to increase its distance from Eric by s, so after the squid's turn, Eric may be as much as $s + d = 2s - 5$ units away. Thus if $n < 2s - 5$, Eric is unable to capture the squid.

1 point for establishing the lower bound.

Proof of lower bound. Since Eric moves at most n times every turn, it suffices to show Eric can win when $n = 2s - 5$. In fact I claim Eric can win on his first move.

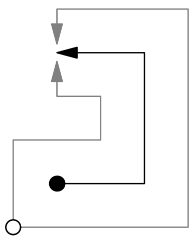


Figure 2: Eric's strategy

Without loss of generality the squid starts at $(0, 0)$. We claim Eric can win on his first turn if he starts from $(-1, -s + 6)$. Call the final position of the squid the *destination*. Assume that the squid only places black ink. This is worse for Eric, and does not affect the squid's first turn.

First note that the squid cannot surround Eric, as that would take at least $s + 1$ moves. Furthermore the squid cannot surround itself, as then it can only perform finitely many moves before losing.

We consider two paths, the "right" and "left" paths, from Eric to the destination; in summary, Eric reaches some point on the squid's path, and moves around the path in the two possible directions until it reaches the destination. We will show at least one of these two paths has length at most $n = 2s - 5$. Let T be the sum of the lengths of the two paths.

1 point for the global idea of attempting to consider the two possible paths that wrap around the squid at once.

Claim. If the squid's ink ever blocks Eric from following the ink, then Eric is able to "jump" to another point on the squid's ink that is closer to the destination, and this decreases the length of Eric's path.

Proof. Without loss of generality, Eric is on the right-hand path. Say a *blockade* is when some part of the ink blocks the square directly to the right of some square on the path. There are two possible blockades, as shown below.

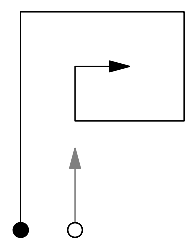


Figure 3: Left-hand blockade

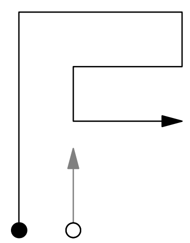


Figure 4: Right-hand blockade

The blockade must be closer to the destination than the current position. Otherwise, we will have already dealt with the intersection before.

In the case of a left-hand blockade, the squid has trapped itself in a closed region, and there are only finitely many squares it can reach. Thus the squid will eventually lose, contradiction. In the case of a right-hand blockade, once Eric reaches the blockade, he can turn right and skip a portion of the blockade. Thus this is in Eric's favor. □

1 point for this claim.

We proceed with the computation. Eric's first step is to reach a point adjacent to some point on the squid's path. (It is possible that the destination is the only possible point Eric can reach in this manner.)

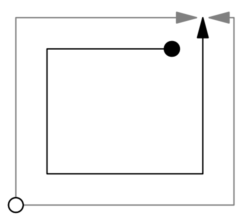


Figure 5: Surrounding the squid ink

It takes $s - 5$ moves to reach a point adjacent to the closest point on the ink path. Once there, we split off into two different directions to surround the squid's path. As shown in

Figure 2 and Figure 5, the union of these two paths (with the first $s-5$ moves omitted) forms a cycle, and $T' = T - 2(s-5)$ is the length of this cycle.

Claim. $T' \leq 2(s+1)$.

Proof. Note that for each corner flanked by two sides of the cycle, the averages of the additional lengths the two surrounding paths gain equals the total length the corner contributes to the length of the squid's ink. Refer to the bottom-right corner in Figure 2.

Thus each unit along the ink corresponds to at most two units along the length of T'. Finally we need to consider the final move from a point adjacent to the destination onto the destination. This yields an additional 2 units, so $T' \leq 2(s+1)$, as claimed. □

2 points for this estimate and the conclusion (shown below) that $T < 2(2s-4)$. If $T < 2(2s-4)$ is not obtained, award 1 point.

By definition, we have $T \leq 2(2s-4)$. Note that if $T < 2(2s-4)$, by Pigeonhole, one of Eric's two choices has length at most $n = 2s-5$, so Eric can win. Assume instead that equality holds.

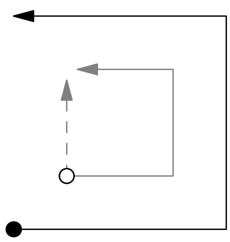


Figure 6: Skipping the U-turn

If there are any *U-turns*, as shown above, Eric can shorten one of the paths. Then the sum of his two choices is now less than $2(2s-4)$, so we may finish as above. Henceforth also assume there are no U-turns. Thus in the worst-case scenario, all of the squid's moves are either north or east.

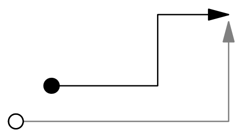


Figure 7: The equality case

If the last move of the squid's turn is to the east, Eric can move all the way to the east, and then up. An analogous argument holds if the last move is to the north. The length of this path is equal to the taxicab distance from Eric to the destination, which is $(s-5)+s = 2s-5 = n$, as desired.

Finally, the answer is $n \geq 2s-5 = 4035$, and we are done.

1 point for resolving the equality case, thus completing the proof.