1^{st} Christmas American Math Olympiad (Day 1)

January 10 to February 3, 2020

CAMO 1. Let $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ (meaning f takes positive real numbers to positive real numbers) be a nonconstant function such that for any positive real numbers x and y,

$$f(x)f(y)f(x+y) = f(x) + f(y) - f(x+y).$$

Prove that there is a constant $a > 1$ such that

$$f(x) = \frac{a^x - 1}{a^x + 1}$$

for all positive real numbers x.

CAMO 2. Let k be a positive integer, $p > 3$ a prime, and n an integer with $0 \leq n \leq p^{k-1}$. Prove that

$$\binom{p^k}{pn} \equiv \binom{p^{k-1}}{n} \pmod{p^{2k+1}}.$$

CAMO 3. Let ABC be a triangle with incircle ω, and let ω touch $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Point M is the midpoint of $\overline{EF}$, and T is the point on ω such that $\overline{DT}$ is a diameter. Line MT meets the line through A parallel to $\overline{BC}$ at P and ω again at Q. Lines DF and DE intersect line AP at X and Y respectively. Prove that the circumcircles of $\triangle APQ$ and $\triangle DXY$ are tangent.

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.

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CAMO 4. Let ABC be a triangle and Q a point on its circumcircle. Let E and F be the reflections of Q over $\overline{AB}$ and $\overline{AC}$, respectively. Select points X and Y on line EF such that $\overline{BX} \parallel \overline{AC}$ and $\overline{CY} \parallel \overline{AB}$, and let M and N be the reflections of X and Y over B and C respectively. Prove that M, Q, N are collinear.

CAMO 5. Let $f(x) = x^2 - 2$. Prove that for all positive integers n, the polynomial

$$P(x) = \underbrace{f(f(\dots f}_{n \text{ times}}(x)\dots)) - x$$

can be factored into two polynomials with integer coefficients and equal degree.

CAMO 6. Let n be a positive integer. Eric and a squid play a turn-based game on an infinite grid of unit squares. Eric's goal is to capture the squid by moving onto the same square as it.

Initially, all the squares are colored white. The squid begins on an arbitrary square in the grid, and Eric chooses a different square to start on. On the squid's turn, it performs the following action exactly 2020 times: it chooses an adjacent unit square that is white, moves onto it, and sprays the previous unit square either black or gray. Once the squid has performed this action 2020 times, all squares colored gray are automatically colored white again, and the squid's turn ends. Moreover, the squid is claustrophobic, so at no point in time is it ever surrounded by a closed loop of black or gray squares. On Eric's turn, he performs the following action at most n times: he chooses an adjacent unit square that is white and moves onto it. Note that the squid can trap Eric in a closed region, so that Eric can never win.

Eric wins if he ever occupies the same square as the squid. Suppose the squid has the first turn, and both Eric and the squid play optimally. Both Eric and the squid always know each other's location and the colors of all the squares. Find all positive integers n such that Eric can win in finitely many moves.

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.