The 3rd CIME I Solutions Pamphlet will be released after the contest.

CONTACT US — *Correspondence about the problems and solutions for this CIME should be sent by PM to:*

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The problems for this CIME were prepared by the CMC's Committee on the CIME.

2020 CAMO/CJMO — The Christmas American Math Olympiad (CAMO) and Christmas Junior Math Olympiad (CJMO) are 6-question, 9-hour, essay-type examinations. The CAMO/CJMO will be held from Friday, January 10, 2020 to Monday, February 3, 2020. Your teacher will not have more details on who qualifies for the CAMO/CJMO in the CMC 10/12 and CIME Teachers' Manuals because we did not make Teachers' Manuals and all students are qualified for the CAMO/CJMO. Students may participate in only one of the CAMO/CJMO, as the problems will intersect. The best way to prepare for the CAMO/CJMO is to study previous years of these exams, which may be found on our website as indicated below.

PUBLICATIONS — For a complete listing of our previous competitions, please visit our website at `http://cmc.ericshen.net/`.

The
Christmas Math Competitions
are made possible by the contributions of following problem-writers and test-solvers:

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MAC CMC
Christmas Math Competitions

Christmas Math Competitions
3rd Annual

CIME I

Christmas Invitational Mathematics Examination I
Friday, January 3, 2020

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU HAVE STARTED YOUR TIMER.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.
3. No aids other than scratch paper, graph paper, rule, compass, and protractor are permitted. **In particular, calculators and computers are not permitted.**
4. A combination of the CIME and the Christmas Math Contest 12 are not used to determine eligibility for participation in the Christmas American Math Olympiad (CAMO). A combination of the CIME and the Christmas Math Contest 10 are not used to determine eligibility for participation in the Christmas Junior Math Olympiad (CJMO). All students are eligible to participate in either than CAMO or the CJMO, but not both, regardless of participation in the Christmas Math Contest 10 or 12. The CAMO and CJMO will be given from FRIDAY, January 10, 2020 to MONDAY, February 3, 2020.
5. You may record your answers on your test booklet, on a separate sheet of paper, or you may simulate contest environments by obtaining an AIME answer form from `https://www.maa.org/math-competitions/aime-archive` and record all of your answers, and certain other information, on the AIME answer form. The answer form will not be collected from you. Only your submission on the CIME Submission Form found at `http://cmc.ericshen.net/CMC-2020` will be graded.

1. A knight begins on the point $(0,0)$ in the coordinate plane. From any point (x,y) the knight moves to either $(x+2,y+1)$ or $(x+1,y+2)$. Find the number of ways the knight can reach the point $(15,15)$.

2. At the local Blast Store, there are sufficiently many items with a price of $\$n.99$ for each nonnegative integer n. A sales tax of 7.5% is applied on all items. If the total cost of a purchase, after tax, is an integer number of cents, find the minimum possible number of items in the purchase.

3. In a math competition, all teams must consist of between 12 and 15 members, inclusive. Mr. Beluhov has $n>0$ students and he realizes that he cannot form teams so that each of his students is on exactly one team. Find the sum of all possible values of n.

4. There exists a unique positive real number x satisfying
$$x=\sqrt{x^2+\frac{1}{x^2}}-\sqrt{x^2-\frac{1}{x^2}}.$$
Given that x can be written as $x=2^{m/n}\cdot 3^{-p/q}$ for positive integers m, n, p, q, with $\gcd(m,n)=\gcd(p,q)=1$, find $m+n+p+q$.

5. Let $ABCD$ be a rectangle with sides $AB>BC$ and let E be the reflection of A over $\overline{BD}$. If $EC=AD$ and the area of $ECBD$ is 144, find the area of $ABCD$.

6. Find the number of complex numbers z satisfying $|z|=1$ and $z^{850}+z^{350}+1=0$.

7. For every positive integer n, define
$$f(n)=\frac{n}{1\cdot 3\cdot 5\cdots(2n+1)}.$$
Suppose that the sum $f(1)+f(2)+\cdots+f(2020)$ can be expressed as $\frac{p}{q}$ for relatively prime integers p and q. Find the remainder when p is divided by 1000.

8. A person has been declared the first to inhabit a certain planet on day $N=0$. For each positive integer $N>0$, if there is a positive number of people on the planet, then either one of the following three occurs, each with probability $\frac{1}{3}$:
   (i) the population stays the same;
   (ii) the population increases by 2^N; or
   (iii) the population decreases by 2^{N-1}. (If there are no greater than 2^{N-1} people on the planet, the population drops to zero, and the process terminates.)

   The probability that at some point there are exactly $2^{20}+2^{19}+2^{10}+2^9+1$ people on the planet can be written as $\frac{p}{3^q}$, where p and q are positive integers such that p is not divisible by 3. Find the remainder when $p+q$ is divided by 1000.

9. Let $ABCD$ be a cyclic quadrilateral with $AB=6$, $AC=8$, $BD=5$, $CD=2$. Let P be the point on $\overline{AD}$ such that $\angle APB=\angle CPD$. Then $\frac{BP}{CP}$ can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

10. Let $1=d_1<d_2<\cdots<d_k=n$ be the positive divisors of a positive integer n. Let S be the sum of all positive integers n satisfying
$$n=d_1^1+d_2^2+d_3^3+d_4^4.$$
Find the remainder when S is divided by 1000.

11. An *excircle* of a triangle is a circle tangent to one of the sides of the triangle and the extensions of the other two sides. Let ABC be a triangle with $\angle ACB=90^\circ$ and let r_A, r_B, r_C denote the radii of the excircles opposite to A, B, C, respectively. If $r_A=9$ and $r_B=11$, then r_C can be expressed in the form $m+\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m+n$.

12. Define a sequence a_0, a_1, a_2, $\ldots$ by
$$a_i=\underbrace{1\ldots1}_{2^i\text{ 1's}}\ \underbrace{0\ldots0}_{(2^i-1)\text{ 0's}}\ 1_2,$$
where a_i is expressed in binary. Let S be the sum of the digits when $a_0a_1a_2\cdots a_{10}$ is expressed in binary. Find the remainder when S is divided by 1000.

13. Chris writes on a piece of paper the positive integers from 1 to 8 in that order. Then, he randomly writes either $+$ or $\times$ between every two adjacent numbers, each with equal probability. The expected value of the expression he writes can be expressed as $\frac{p}{q}$ for relatively prime positive integers p and q. Find the remainder when $p+q$ is divided by 1000.

14. Let ABC be a triangle with sides $AB=5$, $BC=7$, $CA=8$. Denote by O and I the circumcenter and incenter of $\triangle ABC$, respectively. The incircle of $\triangle ABC$ touches $\overline{BC}$ at D, and line OD intersects the circumcircle of $\triangle AID$ again at K. Then the length of DK can be expressed in the form $\frac{m\sqrt{n}}{p}$, where m, n, p are positive integers, m and p are relatively prime, and n is not divisible by the square of any prime. Find $m+n+p$.

15. Find the number of integer sequences a_1, a_2, $\ldots$, a_6 such that
   - $0\le a_1<6$ and $12\le a_6<18$,
   - $1\le a_{k+1}-a_k<6$ for all $1\le k<6$, and
   - there do not exist $1\le i<j\le 6$ such that a_j-a_i is divisible by 6.