2020 CIME I Solutions Document

Christmas Math Competitions

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1. (Answer: 252)

   Say there are A moves of the form $(x, y) \mapsto (x+2, y+1)$ and B moevs of the form $(x, y) \mapsto (x+1, y+2)$. The final position is $(15, 15)$, so $2A + B = A + 2B = 15$. It follows that $A = B = 5$.

   There are 10 moves in total, and 5 are of one type and 5 of the other. All permutations of these 10 moves are valid, so the answer is $\binom{10}{5} = 252$.

2. (Answer: 020)

   Suppose k items were purchased, so that the price before tax is $100N - k$ cents for some integer N. After a tax of 7.5%, i.e. a multiplier of $43/40$ is applied, we need

   $$\frac{43}{40}(100N - k)$$

   to be an integer. It is equivalent for $100N - k$ to be a multiple of 40. Thus k is a multiple of 20, but $k = 20$ is achievable by taking N a sufficiently large odd number.

3. (Answer: 480)

   Suppose t teams are formed. Then the number of students can be in the interval $[12t, 15t]$, thus the set of achievable n is the union of $[12t, 15t]$ for all t.

   Listing the first few sets, we see this set of achievable n begins

   $$[12, 15] \cup [24, 30] \cup [36, 45] \cup [48, 60] \cup [60, 75] \cup [72, 90] \cup \cdots$$

   This motivates the conjecture that all $n \geq 48$ are achievable. Indeed, $n \in [48, 60]$ are all covered by $t = 4$, and for $n \geq 60$, an easy check shows that

   $$\frac{n}{12} - 1 \geq \frac{n}{15} + 1 \implies \left\lfloor \frac{n}{12} \right\rfloor \geq \left\lfloor \frac{n}{15} \right\rfloor.$$

   The requested sum is $(1 + \cdots + 11) + (16 + \cdots + 23) + (31 + \cdots + 35) + (46 + 47) = 480$.

4. (Answer: 014)

   The equation in its given form reveals nothing about x, so we square it:

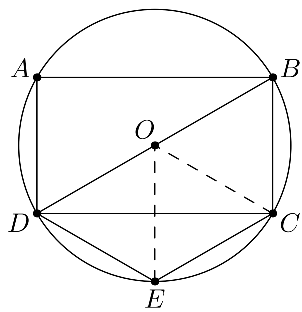
   $$x^2 = 2x^2 - 2\sqrt{x^4 - \frac{1}{x^4}} \implies x^2 = 2\sqrt{x^4 - \frac{1}{x^4}}.$$

Square it again to eliminate the radical: we obtain

$$x^4 = 4\left(x^4 - \frac{1}{x^4}\right) \implies 3x^4 = \frac{4}{x^4} \implies x^8 = \frac{4}{3}.$$

It follows that $x = 2^{1/4} \cdot 3^{-1/8}$, and the requested sum is $1 + 4 + 1 + 8 = 14$.

5. (Answer: 192)



Let O be the center of $ABCD$. Note that $\angle BED = \angle BAD = 90^\circ$, so E also lies on $(ABCD)$. Since BC, CE, ED are all equal, $\angle BOC = \angle COE = \angle EOD = 60^\circ$, so $\triangle OBC$, $\triangle OCE$, $\triangle OED$ are equilateral.

Now $[ECBD] = 3[BOC]$ and $[ABCD] = 4[BOC]$, so

$$[ABCD] = \frac{4}{3}[ECBD] = 192,$$

the answer.

6. (Answer: 100)

Let $a = z^{50}$. Then $a^{17} + a^7 + 1 = 0$ and $|a| = |z|^{50} = 1$. Since a^{17}, a^7, 1 have equal magnitude and sum to 0, their corresponding vectors are sides of an equilateral triangle, thus $\{a^{17}, a^7\} = \{\exp(2\pi i/3), \exp(4\pi i/3)\}$.

There are two cases:

- *Case 1:* $a^7 = \exp(2\pi i/3)$ and $a^{17} = \exp(4\pi i/3)$. Note that

$$\exp(4\pi i/3) = a^{17} = (a^7)^2 \cdot a^3 = \exp(4\pi i/3) \cdot a^3,$$

so a is a third root of unity. It is easy to see that we must have $a = \exp(2\pi i/3)$.

- *Case 2:* $a^7 = \exp(4\pi i/3)$ and $a^{17} = \exp(2\pi i/3)$. Note that

$$\exp(2\pi i/3) = a^{17} = (a^7)^2 \cdot a^3 = \exp(2\pi i/3) \cdot a^3,$$

so a is a third root of unity. It is easy to see that we must have $a = \exp(4\pi i/3)$.

There are two possible values of a, and thus $2 \cdot 50 = 100$ possible values of z.

7. (Answer: 312)

Let $g(n) = f(1) + f(2) + \cdots + f(n)$. I claim that

$$g(n) = \frac{1}{2} - \frac{1}{2 \cdot (2n+1)!!}.$$

To show this, we proceed by induction. The base case $n = 1$ is obvious. If the hypothesis holds for all $k < n$, then

$$\begin{aligned} g(n) &= f(n) + g(n-1) \\ &= \frac{n}{(2n+1)!!} + \left(\frac{1}{2} - \frac{1}{2\cdot(2n-1)!!}\right) \\ &= \frac{1}{2} + \frac{2n}{2\cdot(2n+1)!!} - \frac{2n+1}{2\cdot(2n+1)!!} \\ &= \frac{1}{2} - \frac{1}{2\cdot(2n+1)!!}, \end{aligned}$$

proving the claim. Hence the numerator is $K = \frac{1}{2}(4041!! - 1)$.

We proceed by Chinese Remainder theorem. Modulo 125 is easy: $4041!! \equiv 0 \pmod{125}$, so $K \equiv 62 \pmod{125}$. To evaluate K modulo 8, we need $4041!!$ modulo 16. Considering each residue modulo 16,

$$4041!! \equiv 3^{253}\cdot 5^{253}\cdot 7^{253}\cdot 9^{252}\cdot 11^{252}\cdot 13^{252}\cdot 15^{252} \pmod{16}.$$

Since $\varphi(16) = 8$, by Euler's theorem

$$4041!! \equiv 3\cdot 5\cdot 7\cdot 9 \equiv 1 \pmod{16}.$$

It follows that $K \equiv 0 \pmod 8$, and thus $K \equiv 312 \pmod{1000}$.

8. (Answer: 176)

Solution by Justin Lee Note that after day n,

- the total population is at most $2^{n+1} - 1$, and
- the remainder when the population is divided by 2^n can no longer change after this point.

Henceforth if the population decreases on day $n + 1$, for it to avoid extinction, the population must have been greater than 2^n after day n, so it must have increased on day n.

Consider day n, where $1 \le n < 8$. The population before day n is at most $2^n - 1$. If the population increases by 2^n on day n, then after day $n + 1$, the population must be $1 \mod 2^{n+1}$. Since $2^n <$ day n population $< 2^{n+1}$, we see that on day $n+1$ we cannot increase the population or leave it unchanged. Thus, every increase must be followed by a decrease and every decrease must be preceded by an increase.

Given a sequence of "increase," "decrease," and "neither" such that the above condition is satisfied, note that every (increase, decrease) operation pair leaves the population unchanged, so the population after day 8 is 1 (we cannot increase on day 8, for that forces the population to decrease on day 9 by the modulo restraint on $n = 9$, but then we cannot decrease again on day 10, so the modulo restraint is not satisfied on day 10).

To count the number of such operations for the first 8 days, it suffices to place the decreases amongst 8 slots so that no two decreases are adjacent (this fixes the increases). Letting this be a_8 we have a_6 ways if we decrease on day 8 and a_7 ways otherwise, so we have the recurrence relation $a_n = a_{n-1} + a_{n-2}$. Since $a_1 = 1$ and $a_2 = 2$, we find $a_8 = 34$.

On day 9, the population must increase, for if it remains the same, then after day 10 we cannot have the population $\equiv 2^9 + 1 \mod 2^{10}$; similarly the population must increase on day 10.

Now days 11 through 18 are subject to the same constraints as days 1 through 8, so again, we have 34 ways. Moreover, similarly it follows that the population must have increased on days 19 and 20. Note that if the population ever reaches $2^{20}+2^{19}+2^{10}+2^9+1$, it must do so on day 20. Hence, our probability is $\frac{34^2}{3^{20}}$, and the requested sum is $1156+20=1176$.

9. (Answer: 023)

   Solution by Kaiwen Li Let C' be the reflection of C over $\overline{AD}$, and note that B, P, C' are collinear. Now, if we set $\theta = \angle ABD = \angle AC'D$, then

$$\begin{aligned} BP/CP &= BP/C'P \\ &= [ABD]/[AC'D] \\ &= \left(\frac{1}{2}\cdot AB\cdot BD\cdot \sin\theta\right) \Big/ \left(\frac{1}{2}\cdot AC'\cdot C'D\cdot \sin\theta\right) \\ &= 15/8, \end{aligned}$$

   and the requested sum is $15+8=23$.

10. (Answer: 418)

    First note that n is even; otherwise, n odd implies all four terms on the right-hand side are odd, which is absurd. Thus $d_1=1$ and $d_2=2$. We have three cases: (Henceforth, p and q always denote odd primes.)

    - If $\{d_1,d_2,d_3,d_4\}=\{1,2,p,q\}$, then $1+2^2+p^3+q^4$ is odd, so no solutions.
    - If $\{d_1,d_2,d_3,d_4\}=\{1,2,p,2p\}$, then $p \mid 1+2^2$, so $p=5$. It follows that $n=10130$.
    - If $\{d_1,d_2,d_3,d_4\}=\{1,2,4,p\}$, then $p=3$ gives $n=288$. If $p\geq 5$, then $4p \mid 1+2^2+4^3+p^4$, from which $p \mid 1+2^2+4^3=69$. The latter forces $p=23$, which fails.

    The requested sum is $288+10130=10418$.

11. (Answer: 209)

    Let $a=BC$, $b=CA$, $c=AB$, $s=\frac{a+b+c}{2}$, and K be the area of $\triangle ABC$. Remark that since $\angle ACB=90^\circ$, if the C-excircle touches $\overline{AB},\overline{BC},\overline{CA}$ at C',A',B', respectively, then $CA'I_CB'$ is a square, so $r_C=I_AA'=CA'=s$. It is known that

$$K=r_A(s-a)=r_B(s-b)=r_C(s-c).$$

    Notice that

$$\begin{aligned} s(s-c) &= \frac{(a+b+c)(a+b-c)}{4} = \frac{(a+b)^2-c^2}{4} \\ &= \frac{(a+b)^2-a^2-b^2}{4} = \frac{ab}{2} = K, \end{aligned}$$

    and by Heron's $(s-a)(s-b)=K$ as well. Check that

$$r_A+r_B=\frac{K}{s-a}+\frac{K}{s-b}=K\left(\frac{c}{(s-a)(s-b)}\right)=c.$$

    Furthermore,

$$ab=2K=2\cdot\frac{K}{s-a}\cdot\frac{K}{s-b}=2r_Ar_B.$$

Hence, $a^2 + b^2 = (r_A + r_B)^2$ and $2ab = 4r_Ar_B$. Adding, $a + b = \sqrt{(r_A + r_B)^2 + 4r_Ar_B}$, and it readily follows that

$$r_C = \frac{a+b+c}{2} = \frac{r_A + r_B + \sqrt{(r_A + r_B)^2 + 4r_Ar_B}}{2},$$

which evaluates to $10 + \sqrt{199}$. The requested sum is $10 + 199 = 209$.

12. (Answer: 048)

Recall the identity

$$x^4 + x^2 + 1 = \left(x^2 - x + 1\right)\left(x^2 + x + 1\right).$$

By definition,

$$a_i = 2^{2^{i+1}} - 2^{2^i} + 1.$$

We will compute $7a_0a_1 \cdots a_{10}$. I claim that

$$7\prod_{i=0}^{n} a_i = 2^{2^{i+2}} + 2^{2^{i+1}} + 1.$$

This can be shown using induction. The base case, $n = 0$, is clear, and if the hypothesis holds for $n - 1$, then

$$\begin{aligned} 7\prod_{i=0}^{n} a_i &= a_n \cdot 7\prod_{i=0}^{n-1} a_i \\ &= \left(2^{2^{i+1}} - 2^{2^i} + 1\right)\left(2^{2^{i+1}} + 2^{2^i} + 1\right) \\ &= 2^{2^{i+2}} + 2^{2^{i+1}} + 1, \end{aligned}$$

as claimed.

Note that $1/7 = 0.\overline{001}_2$. I claim the sum of the digits of $\frac{1}{7}\left(2^{2^{12}} + 2^{2^{11}} + 1\right)$ is

$$\left\lfloor \frac{2^{12}}{3} \right\rfloor + \left\lfloor \frac{2^{11}}{3} \right\rfloor + 1.$$

Indeed, the 1's from the first term occupy the kth digits counting from the right, whereas $k \equiv 2^{12} + 1 \pmod 3$, and the 1's from the second term occupy digits with $k \equiv 2^{11} + 1 \pmod 3$. This is the desired answer since no two of $\{2^{12} + 1, 2^{11} + 1, 1\}$ are congreunt modulo 3.

Finally, the answer is $1365 + 682 + 1 = 2048$.

13. (Answer: 225)

Solution by Anthony Wang Let a_n be expected value of the last summand in the expression, and b_n be the expected value of the rest. We wish to find $a_8 + b_8$. It is not hard to see that

$$\begin{aligned} a_n &= \frac{1}{2}n + \frac{1}{2}n \cdot a_{n-1} = \frac{n}{2}(a_{n-1} + 1), \quad \text{and} \\ b_n &= \frac{1}{2}b_{n-1} + \frac{1}{2}(a_{n-1} + b_{n-1}) = b_{n-1} + \frac{1}{2}a_{n-1}. \end{aligned}$$

Since $a_1 = 1$ and $b_1 = 0$, we have $a_2 = 2$, $a_3 = \frac{9}{2}$, $a_4 = 11$, $a_5 = 30$, $a_6 = 93$, $a_7 = 329$, and $a_8 = 1320$, by repeatedly applying the first recurrence. Then

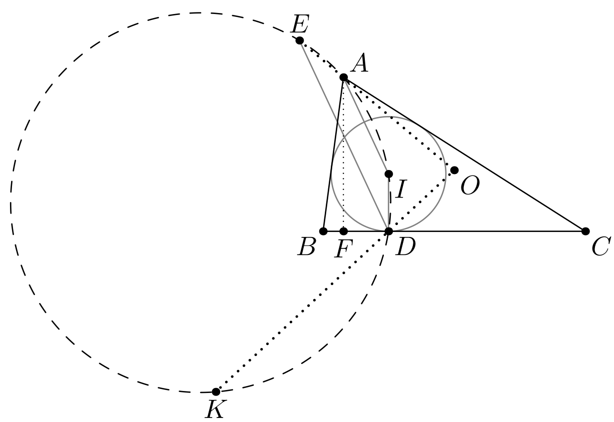
$$b_8 = \frac{1}{2}\left(1 + 2 + \frac{9}{2} + 11 + 30 + 93 + 329\right) = \frac{941}{4},$$

by the second recurrence. Thus

$$a_8 + b_8 = 1320 + \frac{941}{4} = \frac{6221}{4},$$

and the requested sum is $6221 + 4 = 6225$.

14. (Answer: 093)



Let $\overline{AO}$ intersect (AIO) again at E, and let F be the foot from A to $\overline{BC}$. Note that $\overline{AF}$ and $\overline{AO}$ are isogonal wrt. $\angle BAC$, but $\overline{AF} \parallel \overline{ID}$. Consequently,

$$\measuredangle AID = \measuredangle IAF = \measuredangle OAI = \measuredangle EAI,$$

whence $AIDE$ is an isosceles trapezoid. In particular, $AE = ID$.

When $AB = 5$, $BC = 7$, $CA = 8$, we have $\angle A = 60^\circ$, so the area of $\triangle ABC$ is given by $K = 10\sqrt{3}$, the semiperimeter is $s = 10$, the inradius is $r = \sqrt{3}$, and the circumradius is $R = \frac{7}{\sqrt{3}}$. Plugging in the numbers,

$$OD \cdot OK = OA \cdot OE = R(R + r) = \frac{70}{3}.$$

If M denotes the midpoint of $\overline{BC}$, we can compute $BD = s - b = 2$, so $DM = \frac{3}{2}$. Since $\angle BOM = \angle A = 60^\circ$ and $BM = \frac{7}{2}$, we have $OM = \frac{7}{2\sqrt{3}}$. Thus

$$OD = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{7}{2\sqrt{3}}\right)^2} = \frac{\sqrt{57}}{3}.$$

It follows that $OK = \frac{70}{\sqrt{57}}$ and $DK = \frac{17\sqrt{57}}{19}$. The requested sum is $17 + 57 + 19 = 93$.

15. (Answer: 302)

Consider the sequence (b_i) with $0 \le b_i < 6$ and $b_i \equiv a_i \pmod 6$. The number of *mismatches* of (b_i) is the number of i with $b_i > b_{i+1}$.

Note that $\lfloor a_i/6 \rfloor$ increases if and only if (b_i) has a mismatch at index i, so we are counting the number of permutations of $(0, 1, 2, 3, 4, 5)$ with exactly two mismatches.

Let $f(n)$ be the number of permutations of $(0, 1, 2, 3, 4, 5)$ with exactly n mismatches. Note that $f(0) = f(5)$, $f(1) = f(4)$, $f(2) = f(3)$ since reversing add sequence with n mismatches gives a sequence with $5 - n$ mismatches. Hence $f(0) + f(1) + f(2) = 360$.

It is easy to determine $f(0) = 1$, since the only permutation with no mismatches is $(0, 1, 2, 3, 4, 5)$ itself. It suffices to evaluate $f(1)$. For a sequence (b_i) with one mismatch, let S be the set of elements before the mismatch and T the elements after (hence

$|S \cup T| = 6$ and $|S \cap T| = 0$). The elements of S appear in (b_i) in increasing order, and the elements of T also appear in increasing order, so we need to count the number of S determine a sequence with exactly 1 mismatch.

A sequence (b_i) determined by S and T has one mismatch if and only if $\max S > \min T$, as $\max S$ and $\min T$ must be adjacent in (b_i), and between them is where we divided S and T. To count the number of such S, we use complementary counting.

There are a total of $2^6 = 64$ subsets of $\{0, 1, 2, 3, 4, 5\}$. For $\max S < \min T$, S must consist of the first k nonnegative integers for some $0 \le k \le 6$, so 7 subsets are invalid, and the rest work. We conclude $f(1) = 57$.

Finally $f(2) = 360 - 1 - 57 = 302$, and we are done.