The 3rd CIME II Solutions Pamphlet will be released after the contest.

CONTACT US — *Correspondence about the problems and solutions for this CIME should be sent by PM to:*

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The problems for this CIME were prepared by the CMC's Committee on the CIME.

2020 CAMO/CJMO — The Christmas American Math Olympiad (CAMO) and Christmas Junior Math Olympiad (CJMO) are 6-question, 9-hour, essay-type examinations. The CAMO/CJMO will be held from Friday, January 10, 2020 to Monday, February 3, 2020. Your teacher will not have more details on who qualifies for the CAMO/CJMO in the CMC 10/12 and CIME Teachers' Manuals because we did not make Teachers' Manuals and all students are qualified for the CAMO/CJMO. Students may participate in only one of the CAMO/CJMO, as the problems will intersect. The best way to prepare for the CAMO/CJMO is to study previous years of these exams, which may be found on our website as indicated below.

PUBLICATIONS — For a complete listing of our previous competitions, please visit our website at `http://cmc.ericshen.net/`.

The
Christmas Math Competitions
are made possible by the contributions of following problem-writers and test-solvers:

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MAC CMC
Christmas Math Competitions

Christmas Math Competitions
3rd Annual

CIME II

Christmas Invitational Mathematics Examination II
Friday, February 7, 2020

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU HAVE STARTED YOUR TIMER.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.
3. No aids other than scratch paper, graph paper, rule, compass, and protractor are permitted. **In particular, calculators and computers are not permitted.**
4. Figures are not necessarily drawn to scale.
5. A combination of the CIME and the Christmas Math Contest 12 are not used to determine eligibility for participation in the Christmas American Math Olympiad (CAMO). A combination of the CIME and the Christmas Math Contest 10 are not used to determine eligibility for participation in the Christmas Junior Math Olympiad (CJMO). All students are eligible to participate in either than CAMO or the CJMO, but not both, regardless of participation in the Christmas Math Contest 10 or 12. The CAMO and CJMO will be given from FRIDAY, January 10, 2020 to MONDAY, February 3, 2020.
6. You may record your answers on your test booklet, on a separate sheet of paper, or you may simulate contest environments by obtaining an AIME answer form from `https://www.maa.org/math-competitions/aime-archive` and record all of your answers, and certain other information, on the AIME answer form. The answer form will not be collected from you. Only your submission on the CIME Submission Form found at `http://cmc.ericshen.net/CMC-2020` will be graded.

1. Let ABC be a triangle. The bisector of $\angle ABC$ intersects $\overline{AC}$ at E, and the bisector of $\angle ACB$ intersects $\overline{AB}$ at F. If $BF = 1$, $CE = 2$, and $BC = 3$, then the perimeter of $\triangle ABC$ can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

2. Find the number of nonempty subsets S of $\{1, 2, 3, \ldots, 10\}$ such that S has an even number of elements, and the product of the elements of S is even.

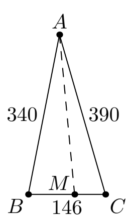
3. In a jar there are blue jelly beans and green jelly beans. Then, 15% of the blue jelly beans are removed and 40% of the green jelly beans are removed. If afterwards the total number of jelly beans is 80% of the original number of jelly beans, then determine the percent of the remaining jelly beans that are blue.

4. The probability a randomly chosen positive integer $N < 1000$ has more digits when written in base 7 than when written in base 8 can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

5. A positive integer n is said to be k-consecutive if it can be written as the sum of k consecutive positive integers. Find the number of positive integers less than 1000 that are either 9-consecutive or 11-consecutive (or both), but not 10-consecutive.

6. An infinite number of buckets, labeled 1, 2, 3, …, lie in a line. A red ball, a green ball, and a blue ball are each tossed into a bucket, such that for each ball, the probability the ball lands in bucket k is 2^{-k}. Given that all three balls land in the same bucket B and that B is even, then the expected value of B can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

7. Let ABC be a triangle with $AB = 340$, $BC = 146$, and $CA = 390$. If M is a point on the interior of segment BC such that the length AM is an integer, then the average of all distinct possible values of AM can be expressed in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.



8. A committee has an oligarchy, consisting of A% of the members of the committee. Suppose that B% of the work is done by the oligarchy. If the average amount of work done by a member of the oligarchy is 16 times the amount of work done by a nonmember of the oligarchy, find the maximum possible value of $B - A$.

9. Let $f(x) = x^2 - 2$. There are N real numbers x such that
$$\underbrace{f(f(\ldots f}_{\text{2019 times}}(x)\ldots)) = \underbrace{f(f(\ldots f}_{\text{2020 times}}(x)\ldots)).$$
Find the remainder when N is divided by 1000.

10. Over all ordered triples of positive integers (a, b, c) for which $a + b + c^2 = abc$, compute the sum of all values of $a^3 + b^2 + c$.

11. Let $ABCD$ be a parallelogram such that $AB = 40$, $BC = 60$, and $BD = 50$. Two externally tangent circles of radius r are positioned in the interior of the parallelogram. The largest possible value of r is $\sqrt{m} - \sqrt{n}$, where m and n are positive integers. Find $m + n$.

12. Positive integers a, b, c satisfy
$$\begin{aligned} \operatorname{lcm}(\gcd(a, b), c) &= 180, \\ \operatorname{lcm}(\gcd(b, c), a) &= 360, \\ \operatorname{lcm}(\gcd(c, a), b) &= 540. \end{aligned}$$
Find the minimum possible value of $a + b + c$.

13. A number is *increasing* if its digits, read from left to right, are strictly increasing. For instance, 5 and 39 are increasing while 224 is not. Find the smallest positive integer not expressible as the sum of three or fewer increasing numbers.

14. A positive integer x is *lexicographically smaller* than a positive integer y if for some positive integer i, the ith digit of x from the left is less than the ith digit of y from the left, but for all positive integers $j < i$, the jth digit of x is equal to the jth digit of y from the left. Say the ith digit of a positive integer with less than i digits is -1. For instance, 11 is lexicographically smaller than 110, which is in turn lexicographically smaller than 12.

    Let A denote the number of positive integers m for which there exists an integer $n \geq 2020$ such that when the elements of the set $\{1, 2, \ldots, n\}$ are sorted lexicographically from least to greatest, m is the 2020th number in this list. Find the remainder when A is divided by 1000.

15. Let $P_1P_2\cdots P_{72}$ be a regular 72-gon with area 1, and let $P_i = P_{i+72}$ for all integers i. Let S be the sum of the squares all positive integers $a < 72$ such that
    - for all i, $P_{i-3a} \neq P_{i+a}$ and $P_{i-a} \neq P_{i+3a}$;
    - for all i, lines $P_{i-3a}P_{i+a}$ and $P_{i-a}P_{i+3a}$ are not parallel, do not coincide, and intersect at a point Q_i; and
    - the points Q_1, Q_2, …, Q_{72} form a polygon with positive, rational area.

    Find the remainder when S divided by 1000.