2020 CIME II Solutions Document

Christmas Math Competitions

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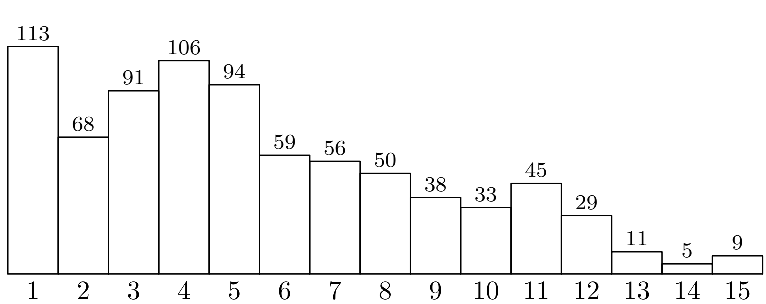
ANSWER KEY

Problem	Author	Answer
1	Eric Shen	067
2	Eric Shen	496
3	Justin Lee	085
4	Nathan Xiong	032
5	Kyle Lee	165
6	Eric Shen	191
7	ES, ES, JL, SL	366
8	J. Lee, E. Shen	060
9	Eric Shen	433
10	F. Clerici, J. Lee	436
11	Kyle Lee	847
12	Sean Li	193
13	Justin Lee	788
14	Justin Lee	443
15	Eric Shen	680

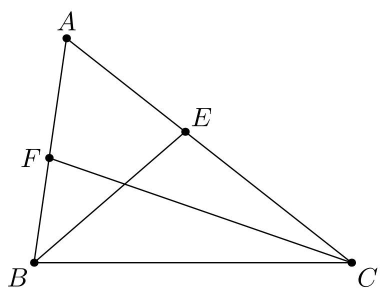
PROBLEM SOLVE-RATE



§1 Solution to CIME II 2020/1

Let ABC be a triangle. The bisector of $\angle ABC$ intersects $\overline{AC}$ at E, and the bisector of $\angle ACB$ intersects $\overline{AB}$ at F. If $BF = 1$, $CE = 2$, and $BC = 3$, then the perimeter of $\triangle ABC$ can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer. 067



Let $AF = x$ and $AE = y$. By the Angle Bisector theorem

$$x = \frac{y+2}{3} \quad \text{and} \quad \frac{y}{2} = \frac{x+1}{3}.$$

Thus $3x = y + 2$ and $3y = 2x + 2$. Substituting $y = 3x - 2$, we have $9x - 6 = 2x + 2$, so $x = 8/7$ and $y = 10/7$. The desired perimeter is $x + y + 6 = 60/7$, and the requested sum is $60 + 7 = 67$.

§2 Solution to CIME II 2020/2

Find the number of nonempty subsets S of $\{1, 2, 3, \dots, 10\}$ such that S has an even number of elements, and the product of the elements of S is even.

Answer. 496

The number of S with an even number of elements is 2^9, by choosing for $i = 1, \dots, 9$ whether i is in S, and whether 10 is in S is dependent on parity.

All S without an even product of elements contains only odd elements. The number of such S is 2^4, by choosing for $i = 1, 3, 5, 7$ whether i is in S, and whether 9 is in S is dependent on parity.

The answer is $2^9 - 2^4 = 496$.

§3 Solution to CIME II 2020/3

In a jar there are blue jelly beans and green jelly beans. Then, 15% of the blue jelly beans are removed and 40% of the green jelly beans are removed. If afterwards the total number of jelly beans is 80% of the original number of jelly beans, then determine the percent of the remaining jelly beans that are blue.

Answer. 085

Let x be the original fraction of blue jellybeans. Counting the number of remaining jelly beans gives

$$\frac{4}{5} = \frac{17}{20}x + \frac{3}{5}(1-x) = \frac{3}{5} + \frac{1}{4}x \implies x = \frac{4}{5}.$$

The number of blue jelly beans remaining is $\frac{17}{20}x = \frac{17}{25}$ of the original jelly beans, and the number of remaining jelly beans is $\frac{4}{5}$ of the original jelly beans, so the fraction of the remaining jelly beans that is blue is

$$\frac{17/25}{4/5} = \frac{17}{20} = 85\%.$$

§4 Solution to CIME II 2020/4

The probability a randomly chosen positive integer $N < 1000$ has more digits when written in base 7 than when written in base 8 can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer. 032

If N has k digits in base 7, then $N \ge 7^k$. It follows that N has less than k digits in base 8, so $N < 8^k$. Hence the number of such N with k digits in base 7 is $8^k - 7^k$.

All such N must be less than 512, so the number of such N is

$$(8^3 - 7^3) + (8^2 - 7^2) + (8^1 - 7^1) + (8^0 - 7^0) = 185.$$

The desired probability is $185/999 = 5/27$, and the requested sum is $5 + 27 = 32$.

§5 Solution to CIME II 2020/5

A positive integer n is said to be k-consecutive if it can be written as the sum of k consecutive positive integers. Find the number of positive integers less than 1000 that are either 9-consecutive or 11-consecutive (or both), but not 10-consecutive.

Answer. 165

For odd k, if n is k-consecutive, then the k consecutive integers have an integer average, so k divides n. Furthermore $n \ge k(k+1)/2$.

For even k, if n is k-consecutive, then the k consecutive integers average to a half-integer, so $k/2$ divides n but k does not divide n. Furthermore $n \ge k(k+1)/2$.

We will complementary count. We begin the answer extraction with PIE:

- The 9-consecutive integers are multiples of 9 from 45 to 999, of which there are 107.
- The 11-consecutive integers are multiples of 11 from 66 to 990, of which there are 85.

- The 9-consecutive and 11-consecutive integers are multiples of 99 from 99 through 990, so there are a total of 10.

Thus $107 + 85 - 10 = 182$ integers are 9-consecutive or 11-consecutive. Now, the overcount:

- The 10-consecutive and 9-consecutive integers are 45 mod 90 from 135 to 945; a total of 10.
- The 10-consecutive and 11-consecutive integers are 55 mod 110 from 165 to 935; a total of 8.
- The only 10-consecutive, 9-consecutive, and 11-consecutive integer is 495.

Hence the overcount is $10 + 8 - 1 = 17$. The answer is $182 - 17 = 165$.

§6 Solution to CIME II 2020/6

An infinite number of buckets, labeled 1, 2, 3, ..., lie in a line. A red ball, a green ball, and a blue ball are each tossed into a bucket, such that for each ball, the probability the ball lands in bucket k is 2^{-k}. Given that all three balls land in the same bucket B and that B is even, then the expected value of B can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer. 191

It is obvious that the probability all three balls land in bucket $2k$ is 64^{-k}. Then, the probability of the given condition occurring is

$$\sum_{k=1}^{\infty} \frac{1}{64^k} = \frac{64^{-1}}{1 - 64^{-1}} = \frac{1}{63}.$$

Hence, the expected value of B is

$$\left(\sum_{k=1}^{\infty} \frac{2k}{64^k} \right) \Big/ \frac{1}{63} = 126 \left(\sum_{j=1}^{\infty} \sum_{k=j}^{\infty} \frac{k}{64^k} \right) = 126 \left(\sum_{j=1}^{\infty} \frac{64^{-j}}{1 - 64^{-1}} \right) = 126 \left(\frac{63^{-1}}{63/64} \right) = \frac{128}{63},$$

and the requested sum is $128 + 63 = 191$.

§7 Solution to CIME II 2020/7

Let ABC be a triangle with $AB = 340$, $BC = 146$, and $CA = 390$. If M is a point on the interior of segment BC such that the length AM is an integer, then the average of all distinct possible values of AM can be expressed in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

Answer. 366

Check that $390^2 - 340^2 = 50 \cdot 730 = 250 \cdot 146 > 146^2$, so $\angle ABC$ is obtuse. It follows that the set of possible values of AM is $(340, 390)$. Its integer elements average to $(340 + 390)/2 = 365$, and the requested sum is $365 + 1 = 366$.

§8 Solution to CIME II 2020/8

A committee has an oligarchy, consisting of $A\%$ of the members of the committee. Suppose that $B\%$ of the work is done by the oligarchy. If the average amount of work done by a member of the oligarchy is 16 times the amount of work done by a nonmember of the oligarchy, find the maximum possible value of $B - A$.

Answer. 060

Scale A and B down by a factor of 100. Assume that the committee has 1 member who can be split into many pieces, and the committee does 1 joule of work. Then

$$16 = \frac{B \text{ joules}}{A \text{ people}} \div \frac{(1-B) \text{ joules}}{(1-A) \text{ people}} = \frac{B}{1-B} \cdot \frac{1-A}{A}.$$

Rearranging,

$$B - AB = 16A - 16AB \implies A = \frac{B}{16 - 15B},$$

so we want to maximize $f(x) = x - \frac{x}{16-15x}$ over $(0, 1)$. Assume the expression equals T. Rearranging,

$$0 = 15x^2 - 15(T+1)x + 16T.$$

To maximize T, we set the discriminant equal to 0: this gives

$$15^2(T+1)^2 = 15 \cdot 64T \implies 0 = T^2 - \frac{34}{15}T + 1.$$

By Po's quadratic method, $T = \frac{17}{15} - \frac{8}{15} = \frac{3}{5}$. The answer is 060.

§9 Solution to CIME II 2020/9

Let $f(x) = x^2 - 2$. There are N real numbers x such that

$$\underbrace{f(f(\dots f}_{2019 \text{ times}}(x)\dots)) = \underbrace{f(f(\dots f}_{2020 \text{ times}}(x)\dots)).$$

Find the remainder when N is divided by 1000.

Answer. 433

The fixed points of f are 2 and -1, so we want $f^{2019}(x) \in \{2, -1\}$. Let a_n be the number of solutions to $f^n(x) \in \{2, -1\}$, and let $b_n = a_n - a_{n-1}$.

By arrows, all x with $f^n(x) \in \{2, -1\}$ for some x must satisfy $x \geq -2$. Otherwise $f(x) > 2$ and f keeps on increasing. Note that for each x_0 such that $f^n(x_0) \in \{2, -1\}$ but $f^{n-1}(x_0) \notin \{2, -1\}$, the solutions to $f(x) = x_0$ satisfy $f^{n+1}(x_0) \in \{2, -1\}$ but $f^n(x_0) \notin \{2, -1\}$. For each $x_0 \notin \{2, -1\}$, $f(x) = x_0$ has two distinct roots.

Hence $b_{n+1} = 2b_n$ for $n \geq 2$. In otherwords, $a_{n+1} = 3a_n - 2a_{n-1}$. By the base case $a_1 = 4$ and $a_2 = 7$, standard methods yield $a_n = 3 \cdot 2^{n-1} + 1$. Hence $N = 3 \cdot 2^{2018} + 1$, and CRT/Euler give an answer of 433.

§10 Solution to CIME II 2020/10

Over all ordered triples of positive integers (a, b, c) for which $a + b + c^2 = abc$, compute the sum of all values of $a^3 + b^2 + c$.

Answer. 436

Rewrite the diophantine as $c^2 - abc + (a + b) = 0$. Thus its roots x, y as a polynomial in c satisfy $x + y = ab$ and $xy = a + b$.

If a, b, x, y are all ≥ 2, then $ab \geq a + b = xy \geq x + y = ab$, so equality holds and $(a, b, x, y) = (2, 2, 2, 2)$. Otherwise without loss of generality $b = 1$, so $a = x + y$ and $a + 1 = xy$. Then $(x - 1)(y - 1) = 2$, so $\{x, y\} = \{2, 3\}$. Then $\{\{a, b\}, \{x, y\}\} = \{\{1, 5\}, \{2, 3\}\}$.

The solution $(a, b, c) = (2, 2, 2)$ contributes 14 to the sum. For the rest of the solutions, a, b, c are all each of $\{1, 2, 3, 5\}$ exactly 2 times, so the requested sum is

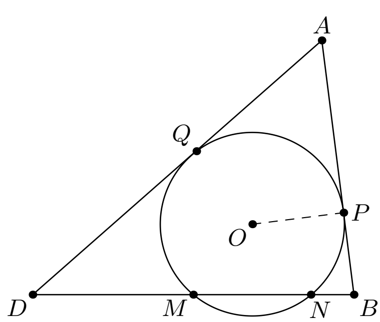
$$14 + 2\left(1^3 + 2^3 + 3^3 + 5^3\right) + 2\left(1^2 + 2^2 + 3^2 + 5^2\right) + 2(1 + 2 + 3 + 5) = 436.$$

§11 Solution to CIME II 2020/11

Let $ABCD$ be a parallelogram such that $AB = 40$, $BC = 60$, and $BD = 50$. Two externally tangent circles of radius r are positioned in the interior of the parallelogram. The largest possible value of r is $\sqrt{m} - \sqrt{n}$, where m and n are positive integers. Find $m + n$.

Answer. 847

Solution by Kaiwen Li Point labels should be evident from the digram below. Note that M is the midpoint of $\overline{BD}$, which is true by symmetry in a parallelogram.



Scale the diagram down by a factor of 5, and denote by x the new length of AP. By power of a point,

$$\begin{aligned}(12 - x)^2 + (8 - x)^2 &= 5(5 - MN) + 5(5 + MN) \\ &= 50,\end{aligned}$$

so $x = 10 - \sqrt{21}$. Now, compute $\tan \frac{A}{2} = \frac{\sqrt{7}}{5}$: it then follows that our desired radius (scaled back up) is

$$r_M = 5x \cdot \tan \frac{A}{2} = 10\sqrt{7} - 7\sqrt{3}.$$

The requested sum is $700 + 147 = 847$.

§12 Solution to CIME II 2020/12

Positive integers a, b, c satisfy

$$\begin{aligned}\operatorname{lcm}(\gcd(a,b),c) &= 180,\\ \operatorname{lcm}(\gcd(b,c),a) &= 360,\\ \operatorname{lcm}(\gcd(c,a),b) &= 540.\end{aligned}$$

Find the minimum possible value of $a+b+c$.

Answer. 193

Solution by Justin Lee (unedited) Note that $180 = 2^2 \cdot 3^2 \cdot 5, 360 = 2^3 \cdot 3^2 \cdot 5, 540 = 2^2 \cdot 3^3 \cdot 5$.

Looking at the powers of two in a, b, c, if these values are e_1, e_2, e_3, then we look at $\max(\min(e_1, e_2), e_3)$ (and the cyclic shifts), which are either the "middle number" (i.e., the number not the largest or smallest amongst the three) or the largest number. Hence we have $e_2 = 3$ and $\max(e_1, e_3) = 2$.

Similarly, looking at powers of three gives $v_3(c) = 3$ and $\max(v_3(a), v_3(b)) = 2$.

Now the powers of five give that two out of the three values $v_5(a), v_5(b), v_5(c)$ equal 1.

We see that the optimal is $5 \cdot 9 + 8 \cdot 5 + 27 \cdot 4 = \boxed{193}$.

§13 Solution to CIME II 2020/13

A number is *increasing* if its digits, read from left to right, are strictly increasing. For instance, 5 and 39 are increasing while 224 is not. Find the smallest positive integer not expressible as the sum of three or fewer increasing numbers.

Answer. 788

It is easy to construct all integers less than 99 as the sum of at most two increasing integers. Say, if we choose $\overline{ab}$ with $a > 0$ $b < 9$ (or the number itself is increasing), then select $10(a-1)+9$ and $b+1$. Furthermore all single-digit integers are increasing.

Now for $a \le 7$ and $\overline{bc} \neq 99$, consider $\overline{a89} + \overline{bc}$. All such numbers are attained since $\overline{bc}$ is the sum of at most two increasing integers. This attains all numbers less than $789 + 99$ not $88 \pmod{100}$. Furthermore $\overline{a79} + 9$ attains all numbers at most 688 that are $88 \pmod{100}$.

Thus all numbers less than 788 are attainable. Suppose 788 is the sum of three increasing numbers, and the sum of the units digits is c, tens digits b, and hundreds digits a. Then $c \le 27$, so $c \in \{8, 18\}$.

- If $c = 8$, then $b \in \{8, 18\}$. But $b < c$, contradiction.
- If $c = 18$, then $b \in \{7, 17\}$. But $b \le c - 2$ unless two of the three integers are less than 10 (this case is easy to outrule), so $b = 7$ and thus $a = 7$. This fails since $a < b$.

The answer is 788.

§14 Solution to CIME II 2020/14

A positive integer x is *lexicographically smaller* than a positive integer y if for some positive integer i, the ith digit of x from the left is less than the ith digit of y from the left, but for all positive integers $j < i$, the jth digit of x is equal to the jth digit of y from the left. Say the ith digit of a positive integer with less than i digits is -1. For instance, 11 is lexicographically smaller than 110, which is in turn lexicographically smaller than 12.

Let A denote the number of positive integers m for which there exists an integer $n \geq 2020$ such that when the elements of the set $\{1, 2, \ldots, n\}$ are sorted lexicographically from least to greatest, m is the 2020th number in this list. Find the remainder when A is divided by 1000.

Answer. 443

Solution by Justin Lee (unedited) We claim that a number m satisfies the condition if $m \neq 10^k$ where $0 \leqslant k \leqslant 2018$ and there are fewer than 2020 numbers that are less than m and lexicographically smaller than m.

Indeed, consider the function $s(n)$, which counts the number of numbers less than n and lexicographically less than m, where n from m to 10^{2019}. Note that $s(m) \leqslant 2019$ and $s\left(10^{2019}\right) > 2019$ (because $10^0, 10^1, \ldots, 10^{2018}$ are lexicographically less than m), so by continuity, we arrive at the desired claim. Now define $f(n)$ to be the number of good numbers, i.e., numbers less than or equal to n and lexicographically less than or equal to n (we include equality for sake of simplicity). We wish to count the number of numbers n for which $f(n) \leqslant 2020$. Let $n = \sum_{i=0}^{k} a_i \cdot 10^i = \overline{a_k a_{k-1} \ldots a_0}$; the number of good $k+1$-digit numbers is $n - 10^k + 1$, the number of good k-digit numbers is $\left\lfloor \frac{n}{10} \right\rfloor - 10^{k-1} + 1$, and so on so forth. Hence,

$$f(n) = \sum_{i=0}^{k} \left(\left\lfloor \frac{n}{10^i} \right\rfloor - 10^{k-i} + 1 \right)$$

When $k > 3$, note that if $a_k > 1$ then $f(n) > n - 10^k > 2020$. Similarly, if $a_{k-i} > 0$ for some $0 < i \leqslant k-3$ then similarly $f(n) > n - 10^k \geqslant 10^{k-i} > 2020$. Thus, $n = 10^k + x$ for some $x < 10^4$.

For each possible value of x, we shall count the number of possible $k > 3$ such that $f(10^k + x) \leqslant 2020$. Letting $x = \overline{abcd}$, we see that $f(10^k + x) = (k+1) + g(x)$ where $g(x) = \overline{abcd} + \overline{abc} + \overline{ab} + a$. We need $g(x) < 2016$ and the number of possible values of k for each value of x is $2016 - g(x)$. Since $g(x)$ is an increasing function, this implies $x \leqslant 1815$. As we sum over all values of x, we obtain

$$\begin{aligned}
\sum_{x=1}^{1815} (2016 - g(x)) &= \sum_{x=1}^{1815} \left(2016 - x - \left\lfloor \frac{x}{10} \right\rfloor - \left\lfloor \frac{x}{100} \right\rfloor - \left\lfloor \frac{x}{1000} \right\rfloor \right) \\
&= 2016 \cdot 1815 - \left(\frac{1815 \cdot 1816}{2} + 10 \cdot \frac{180 \cdot 181}{2} \right. \\
&\qquad \left. + 6 \cdot 181 + 100 \cdot \frac{17 \cdot 18}{2} + 16 \cdot 18 + 1 \cdot 816 \right) \\
&= 1830630
\end{aligned}$$

Now it remains to count the number of such $n < 10000$ with $f(n) \leqslant 2020$. It follows that all $n \leqslant 2816$ work. Thus, excluding $n = 1, 10, 100, 1000$ and including $n = 10^{2019}$, our answer is $1830630 + 2816 - 4 + 1 \equiv \boxed{443} \pmod{1000}$.

§15 Solution to CIME II 2020/15

Let $P_1P_2 \cdots P_{72}$ be a regular dodecagon with area 1, and let $P_i = P_{i+72}$ for all integers i. Let S be the sum of the squares all positive integers $a < 72$ such that

- for all i, $P_{i-3a} \neq P_{i+a}$ and $P_{i-a} \neq P_{i+3a}$;
- for all i, lines $P_{i-3a}P_{i+a}$ and $P_{i-a}P_{i+3a}$ are not parallel, do not coincide, and intersect at a point Q_i; and
- the points Q_1, Q_2, …, Q_{72} form a polygon with positive, rational area.

Find the remainder when S is divided by 1000.

Answer. 680

Toss on the complex plane, and assume $P_0 = 1$ and the center of the polygon is 0. It suffices for OQ_i^2 to be rational. We find all angles θ such that if $\omega = e^{\theta i}$, then the intersection q of the line through ω^3 and ω^{-1} and real axis is the square root of a rational number.

Then q is the intersection of $\overline{\omega^3\omega^{-1}}$ and $\overline{(1)(-1)}$, so by the complex chord intersection formula,

$$q = \frac{\omega^3 \cdot \omega^{-1}(1 + (-1)) - 1(-1)(\omega^3 + \omega^{-1})}{\omega^3 \cdot \omega^{-1} - 1(-1)} = \frac{\omega^3 + \omega^{-1}}{\omega^2 + 1}.$$

However note that

$$\begin{aligned} \left|\omega^3 + \omega^{-1}\right|^2 &= \left[\cos(3\theta) + \cos(-\theta)\right]^2 + \left[\sin(3\theta) + \sin(-\theta)\right]^2 \\ &= 2 + 2\left[\cos(3\theta)\cos(-\theta) + \sin(3\theta)\sin(-\theta)\right] \\ &= 2 + 2\cos(4\theta), \end{aligned}$$

and similarly

$$\begin{aligned} \left|\omega^2 + 1\right|^2 &= [\cos(2\theta) + 1]^2 + \sin^2(2\theta) \\ &= 2 + 2\cos(2\theta). \end{aligned}$$

Since q is real,

$$q^2 = \frac{2 + 2\cos(4\theta)}{2 + 2\cos(2\theta)} = \frac{\cos^2 2\theta}{\cos^2\theta}$$

needs to be rational. Note that

$$|q| = \frac{\cos 2\theta}{\cos\theta} = 2\cos\theta - \frac{1}{\cos\theta} \implies 0 = 2\cos^2\theta - |q|\cos\theta - 1,$$

so since $q^2 \in \mathbb{Q}$, $\cos\theta$ is the root of a polynomial with rational coefficients and degree 4.

We turn to the following well-known lemma:

Lemma (Minimal polynomial of $\cos\theta$)

Let $\theta = \frac{2\pi k}{n}$, where k and n are relatively prime. Then the minimal polynomial of $\cos\theta$ over $\mathbb{Q}$ has degree $\frac{\varphi(n)}{2}$.

Thus the minimal polynomial of $\cos\theta$ over $\mathbb{R}$ has degree at most $\frac{\varphi(n)}{2}$, so we have $\frac{\varphi(n)}{2} \le 4$, or $\varphi(n) \le 8$.

Claim (Extracting n). If $\varphi(n) \le 8$ and $n \mid 72$, then $n \in \{2, 3, 4, 6, 8, 9, 12, 18, 24\}$.

Proof. (Details omitted.) The largest n with $\varphi(n) \le 8$ is 30, and an exhaustive check gives the above list. □

Now all that remains is answer extraction. We will find all $\theta < \pi$, since θ works if and only if $2\pi - \theta$ works, and $\theta = \pi$ violates the first condition.

- We first take care of $n \mid 12$, so $\theta = \frac{\pi k}{6}$ for some $0 < k < 6$. Of these, $k \in \{1, 2, 4, 5\}$ work, giving $\frac{1}{3}$, 1, 1, $\frac{1}{3}$, respectively.
- Let $n = 8$, so $\theta = \frac{\pi k}{4}$ for $k \in \{1, 3\}$. These both give zero area, thus they don't work.
- Let $n = 9$ or $n = 18$, so $\theta = \frac{\pi k}{9}$ for $k \in \{1, 2, 4, 5, 7, 8\}$. These all give irrational area.
- Let $n = 24$, so $\theta = \frac{\pi k}{24}$ for $k \in \{1, 5, 7, 11\}$. These all give irrational area.

Hence the θ that work are $\pm\frac{\pi}{6}, \pm\frac{2\pi}{6}, \pm\frac{4\pi}{6}, \pm\frac{5\pi}{6}$, which give $\frac{a}{6} \in \{1, 2, 4, 5, 7, 8, 10, 11\}$. It follows that the sum of the squares of all possible values of $\frac{a}{6}$ is

$$\sum \frac{a^2}{36} = 1^2 + 2^2 + 4^2 + 5^2 + 7^2 + 8^2 + 10^2 + 11^2 = 380,$$

from which $S = 36 \cdot 380 = 13680$, and the requested remainder is 680.