3^{rd} Christmas Junior Math Olympiad (Day 1)

January 10 to February 3, 2020

CJMO 1. Let N be a positive integer, and let S be the set of all tuples with positive integer elements and a sum of N. For instance, $t_1 = (N)$, $t_2 = (1, 1, N-2)$, $t_3 = (1, N-1)$, and $t_4 = (N-1, 1)$ are all distinct tuples in S. For all tuples t, let $p(t)$ denote the product of all the elements of t. For instance, $p(t_1) = N$, $p(t_2) = N-2$, and $p(t_3) = p(t_4) = N-1$.

Evaluate the expression (where we sum over all elements t of S)

$$\sum_{t \in S} p(t).$$

CJMO 2. Let $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ (meaning f takes positive real numbers to positive real numbers) be a nonconstant function such that for any positive real numbers x and y,

$$f(x)f(y)f(x+y) = f(x) + f(y) - f(x+y).$$

Prove that there is a constant $a > 1$ such that

$$f(x) = \frac{a^x - 1}{a^x + 1}$$

for all positive real numbers x.

CJMO 3. Let ABC be an acute triangle with circumcenter O, orthocenter H, and $\angle A = 45^\circ$. Denote by M the midpoint of $\overline{BC}$, and let P be a point such that $\overline{AP}$ is parallel to $\overline{BC}$ and $\angle HMB = \angle PMC$. Show that if segment OP intersects the circle with diameter $\overline{AH}$ at Q, then $\overline{OA}$ is tangent to the circumcircle of $\triangle APQ$.

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.

3^{rd} Christmas Junior Math Olympiad (Day 2)

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CJMO 4. For all positive integers k, define $s(k)$ to be the result when the last digit of k is moved to the front of k. For instance, $s(2020) = 202$ and $s(1234) = 4123$. For each positive integer n, find the number of positive integers $k < 10^n$ that satisfy $s(9k) = 9s(k)$.

CJMO 5. Let ABC be a triangle, and D be a point on the internal angle bisector of $\angle BAC$ but not on the circumcircle of $\triangle ABC$. Suppose that the circumcircle of $\triangle ABD$ intersects $\overline{AC}$ again at P and the circumcircle of $\triangle ACD$ intersects $\overline{AB}$ again at Q. Denote by O_1 and O_2 the circumcenters of $\triangle ABD$ and $\triangle ACD$, respectively. Prove that the circumcenters of $\triangle ABC$, $\triangle APQ$, and $\triangle AO_1O_2$ are collinear.

CJMO 6. Let $f(x) = x^2 - 2$. Prove that for all positive integers n, the polynomial

$$P(x) = \underbrace{f(f(\ldots f}_{n \text{ times}}(x)\ldots)) - x$$

can be factored into two polynomials with integer coefficients and equal degree.

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.