2020

CMC 10A

DO NOT OPEN UNTIL FRIDAY, December 27, 2019

Christmas Math Competitions

Questions and comments about problems and solutions for this exam should be sent by PM to:

AOPS12142015, **FedeX333X**, and **TheUltimate123**.

The 3rd Annual CIME will be held on Friday, January 3, 2020, with the alternate on Friday, February 7, 2020. It is a 15-question, 3-hour, integer-answer exam. You will be invited to participate regardless of your score on this competition. All students will be invited to take either the 1st Annual Christmas American Math Olympiad (CAMO) or the 3rd Christmas Junior Math Olympiad (CJMO) on Friday, January 10, 2020.

A complete listing of our previous publications may be found at our web site:

`http://cmc.ericshen.net/`

****Administration On An Earlier Date Will Literally Be Impossible****

1. All the information needed to administer this exam is contained in the non-existent CMC 10/12 Teacher's Manual. PLEASE READ THE MANUAL EVERY DAY BEFORE DECEMBER 27, 2019.
2. YOU must not verify on the CMC 10/12 COMPETITION CERTIFICATION FORM (found on `maa.org/amc` under "AMC 10A") that you followed all rules associated with the administration of the exam.
3. If you chose to obtain an AMC 10 Answer Sheet from the MAA's website, it must be returned to yourself the day after the competition. Ship with inappropriate postage without using a tracking method. FedeX333X or UPS is strongly recommended.
4. The publication, asexual reproduction, sexual reproduction, or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the definite (but not indefinite) integrity of the results. Dissemination via phone, email, raven, or digital media of any type during this period is a violation of the competition rules.

The **Christmas Math Competitions** *is made possible by the contributions of the following problem-writers and test-solvers:*

David Altizio, Allen Baranov, Luke Choi, Federico Clerici, Mason Fang, Raymond Feng, Preston Fu, Justin Lee, Kyle Lee, Kaiwen Li, Sean Li, Eric Shen, Anthony Wang, Andrew Wen, Nathan Xiong, *and* Joseph Zhang

MAC CMC
Christmas Math Competitions

Christmas Math Competitions
3rd Annual

CMC 10A

Christmas Math Contest 10A
Friday, December 27, 2019

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU HAVE STARTED YOUR TIMER.
2. This is a 25-question multiple-choice exam. Each question is followed by answers marked A, B, C, D, and E. Only one of these is correct.
3. Mark your answer to each problem however you want. If you would like to create a more realistic test experience, then you may obtain an AMC 10 Answer Sheet from `https://www.maa.org/math-competitions/amc-10-12/` and mark you answer to each problem on the AMC 10 Answer Sheet with a #2 pencil. To simulate the real test, check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer sheet will be graded in a real test. For the CMC, **you must submit your answers using the Submission Form found at `http://cmc.ericshen.net/CMC-2020/`. Only answers submitted to the Submission Form will be scored.**
4. Scoring: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. Only scratch paper, graph paper, rulers, compasses, protractors, and erasers are allowed as aids. No calculators, smartwatches, phones, computing devices, or resources such as Wolfram Alpha are allowed. No problems on the exam require the use of a calculator.
6. Figures are not necessarily not drawn to scale.
7. Before beginning the exam, you will ask yourself to record certain information on the answer form if you chose to obtain an AMC 10 Answer Sheet from `https://www.maa.org/math-competitions/amc-10-12/`.
8. When you give yourself the signal, begin working on the problems. You will have **75 minutes** to complete the exam.
9. When you finish the exam, *sign your name* in the space provided at the top of the Answer Sheet should you choose to obtain one from `https://www.maa.org/math-competitions/amc-10-12/`.

The Committee on the Christmas Math Competitions reserves the right to disqualify scores from an individual if it determines that the required security procedures were not followed.

All students will be invited to take the 3rd annual Christmas Invitational Math Examination (CIME) on Friday, January 3, 2020 and Friday, February 7, 2020. More details about the CIME are in the back of this test booklet.

1. What is the value of

$$\frac{20^{2+0!}}{2+0+2+0}+20^{2^0}?$$

(A) 2018 **(B)** 2019 **(C)** 2020 **(D)** 2021 **(E)** 2169

2. For how many integers n does $2^{2^n} = 2^{n^2}$ hold?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

3. Jerry is at the gym and is going to use the bench press. He wants to put on 5, 5, 10, 10, 10, and 20 pound weights on either side of the bar such that the bar is balanced. If equal weights are indistinguishable, the two sides of the bar are distinguishable, he must use all six weights, and order doesn't matter, how many ways can he put on these weights?

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

4. Let $f(n) = n^2+n+2020$. What is the sum of the distinct prime factors of the number $f(2019)$?

(A) 90 **(B)** 108 **(C)** 110 **(D)** 111 **(E)** 676

5. A regular hexagon with side length 2 is positioned in the coordinate plane such that it has one vertex at $(3, 0)$ and another at $(-1, 0)$. What is the ratio of the area of the portion of the hexagon to the right of the y-axis to the area of the portion to the left?

(A) $2\sqrt{3}$ **(B)** 4 **(C)** 5 **(D)** $3\sqrt{3}$ **(E)** 6

6. Let n be a positive integer, and consider the set

$$S = \{1, 4, 9, 16, 25, 36, n\}$$

where all seven of its elements are distinct. We replace every odd element k of S with $k+1$, and we replace every even element k of S with $k-1$. It is given that the median of the resulting set is equal to the median of S. How many possible values of n are there?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 6

7. Let a_0, a_1, $\ldots$, a_{2019} be real numbers such that $a_0 = 0$ and $a_n + a_{n+1}^{-1} = 2$ for every integer $n \geq 0$. What is the value of the product $a_1a_2\cdots a_{2019}$?

(A) $\frac{1}{2020}$ **(B)** $\frac{1}{2019}$ **(C)** $\frac{1}{1010}$ **(D)** $\frac{1}{1009}$ **(E)** 1

8. A circle of radius r centered at the origin of the coordinate plane intersects the graph of $x^2 + y = x + y^2$ at exactly two points. What is the maximum possible value of r?

(A) $\frac{1}{2}$ **(B)** $\frac{1}{\sqrt{2}}$ **(C)** 1 **(D)** $\sqrt{2}$ **(E)** 2

9. Tasty and Stacy are playing a game. Tasty rolls a fair 6-sided die and Stacy rolls a fair n sided die, whose faces are numbered from 1 to n. Stacy wins if the sum of the values of the two dice is divisible by 5. For which of the following values of n does Stacy maximize her chances to win?

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

10. How many subsets S of the following four are there such that it is possible for all statements in S to be true and all other statements to be false?

- At least one of these statements is true.
- Both this statement and the following statement are true.
- This statement is true.
- The first statement is false.

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

11. Positive integers b_4 and b_6 have the property that there is a unique positive integer b_5 such that

$$1 > \frac{b_4}{4^2} > \frac{b_5}{5^2} > \frac{b_6}{6^2}.$$

If $b_4 + b_6 = 24$, what is the unique value of b_5?

(A) 9 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 13

12. Let $P(x)$ be a polynomial satisfying $P(1+2^n) = 3+4^n$ for all integers n. What is the value of $P(0)$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

13. There exists a positive integer n such that

$$\sqrt{n+\sqrt{n+\sqrt{n+\cdots}}} = \frac{1000}{n+\frac{1000}{n+\frac{1000}{n+\cdots}}}.$$

What is the sum of the digits of n?

(A) 1 **(B)** 2 **(C)** 9 **(D)** 10 **(E)** 18

14. Let $ABCD$ be a rectangle. Suppose the vertices of $ABCD$ have x-coordinates of 1, 2, 3, 4 in some order. What is the minimum possible area of $ABCD$?

(A) 2 **(B)** 4 **(C)** $\frac{9}{2}$ **(D)** 8 **(E)** 9

15. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. What is the sum of the squares of the real numbers x for which $x^2 - 20\lfloor x \rfloor + 19 = 0$?

(A) 362 **(B)** 702 **(C)** 703 **(D)** 1024 **(E)** 1405

16. Let P be a point in the interior of square $ABCD$ so that the ratio of the areas of $\triangle ACP$ and $\triangle BDP$ is $2:1$ and the ratio of the areas of $\triangle ABP$ and $\triangle CDP$ is $3:1$. What is the ratio of the areas of $\triangle ADP$ and $\triangle BCP$?

(A) $\frac{1}{6}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{2}{7}$ **(D)** $\frac{5}{12}$ **(E)** $\frac{5}{7}$

17. We say that a parallelogram is *non-tilted* if it has at least one pair of opposite sides parallel to one of the coordinate axes. How many non-tilted parallelograms are there whose vertices are points (a, b) with integer coordinates $0 \le a \le 4$ and $0 \le b \le 4$?

(A) 300 **(B)** 400 **(C)** 500 **(D)** 550 **(E)** 600

18. For a set of positive integers S, its *primality* is the number of primes that divide some element of S. For instance, $\{1\}$ has primality 0 and $\{12, 21\}$ has primality 3. If a set S of n consecutive integers has primality 10, what is the maximum possible value of n?

(A) 1 **(B)** 10 **(C)** 16 **(D)** 30 **(E)** 46

19. Let ABC be a triangle with $AB = 30$, $BC = 51$, $CA = 63$. Points P and Q lie on $\overline{BC}$, R lies on $\overline{CA}$, and S lies on $\overline{AB}$ such that $PQRS$ is a parallelogram, and the center of $PQRS$ coincides with the centroid of $\triangle ABC$. What is the area of parallelogram $PQRS$?

(A) 84 **(B)** 126 **(C)** 168 **(D)** 336 **(E)** 378

20. Over all $N > 0$, what is the maximum possible number of permutations a, b, c, d, e of the integers 2, 3, 5, 7, 11 such that it is possible to put parentheses around the expression

$$a \div b \div c \div d \div e$$

so that it equals N?

(A) 6 **(B)** 24 **(C)** 36 **(D)** 60 **(E)** 120

21. Let acute $\triangle ABC$ have circumradius of length 10. Suppose D is the foot of the altitude from A to BC such that $BD = 4$ and $CD = 8$. The area of $\triangle ABC$ can be written as $a + b\sqrt{c}$ for positive integers a, b, c, where c is not divisible by the square of any prime. What is the value of $a + b + c$?

(A) 72 **(B)** 74 **(C)** 76 **(D)** 78 **(E)** 80

22. Mr. Green gives his large class of 2020 students a test with n problems. For each problem, a student receives p points, where $p \in \{1, 2, n\}$. After grading all the students' papers, he realizes that every student has a unique total number of points. What is the smallest possible value of n?

(A) 45 **(B)** 46 **(C)** 62 **(D)** 63 **(E)** 64

23. Let A, B, C, and D be four points in space such that

$$\angle BAC = \angle CAD = \angle DAB = 60^\circ.$$

If $AB = 1$, $AC = 2$, and $AD = 6$, then what is the square of the distance between A and the plane of $\triangle BCD$?

(A) $\frac{2}{3}$ **(B)** $\frac{6}{7}$ **(C)** $\frac{13}{14}$ **(D)** $\frac{15}{16}$ **(E)** 1

24. How many ordered triples (a, b, c) of not necessarily distinct positive integer divisors of $216{,}000$ satisfy

$$\operatorname{lcm}(a, \gcd(b, c)) = \operatorname{lcm}(b, \gcd(a, c)) = \operatorname{lcm}(c, \gcd(a, b))?$$

(A) $33{,}880$ **(B)** $81{,}200$ **(C)** $106{,}400$ **(D)** $120{,}960$ **(E)** $212{,}800$

25. Call a four-digit positive integer *asuboptimal* if the sum of two of its digits is equal to the sum of the other two digits. For instance, 1234, 2020, and 9801 are all asuboptimal. How many four-digit positive integers are asuboptimal?

(A) 1494 **(B)** 1557 **(C)** 1584 **(D)** 1632 **(E)** 1845