2020

CMC 10B

DO NOT OPEN UNTIL SUNDAY, January 26, 2020

Christmas Math Competitions

Questions and comments about problems and solutions for this exam should be sent by PM to:

AOPS12142015, **FedeX333X**, and **TheUltimate123**.

The 3rd Annual CIME will be held on Friday, January 3, 2020, with the alternate on Friday, February 7, 2020. It is a 15-question, 3-hour, integer-answer exam. You will be invited to participate regardless of your score on this competition. All students will be invited to take the 3rd Annual Christmas American Math Olympiad (CAMO) or the Christmas Junior Math Olympiad (CJMO) on Friday, January 10, 2020.

A complete listing of our previous publications may be found at our web site:

`http://cmc.ericshen.net/`

****Administration On An Earlier Date Will Literally Be Impossible****

1. All the information needed to administer this exam is contained in the non-existent CMC 10/12 Teacher's Manual. PLEASE READ THE MANUAL EVERY DAY BEFORE JANUARY 26, 2020.
2. YOU must not verify on the CMC 10/12 COMPETITION CERTIFICATION FORM (found on `maa.org/amc` under "AMC 10B") that you followed all rules associated with the administration of the exam.
3. If you chose to obtain an AMC 10 Answer Sheet from the MAA's website, it must be returned to yourself the day after the competition. Ship with inappropriate postage without using a tracking method. FedeX333X or UPS is strongly recommended.
4. The publication, asexual reproduction, sexual reproduction, or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the definite (but not indefinite) integrity of the results. Dissemination via phone, email, raven, or digital media of any type during this period is a violation of the competition rules.

The **Christmas Math Competitions**
is made possible by the contributions of the following problem-writers and test-solvers:

David Altizio, Allen Baranov, Luke Choi,
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Preston Fu, Justin Lee, Kyle Lee, Kaiwen Li,
Sean Li, Eric Shen, Anthony Wang, Andrew Wen,
Nathan Xiong, *and* Joseph Zhang

MAC CMC
Christmas Math Competitions

Christmas Math Competitions
3rd Annual

CMC 10B

Christmas Math Contest 10B
Sunday, January 26, 2020

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU HAVE STARTED YOUR TIMER.
2. This is a 25-question multiple-choice exam. Each question is followed by answers marked A, B, C, D, and E. Only one of these is correct.
3. Mark your answer to each problem however you want. If you would like to create a more realistic test experience, then you may obtain an AMC 10 Answer Sheet from `https://www.maa.org/math-competitions/amc-10-12/` and mark you answer to each problem on the AMC 10 Answer Sheet with a #2 pencil. To simulate the real test, check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer sheet will be graded in a real test. For the CMC, **you must submit your answers using the Submission Form found at `http://cmc.ericshen.net/CMC-2020/`. Only answers submitted to the Submission Form will be scored.**
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. Only scratch paper, graph paper, rulers, compasses, protractors, and erasers are allowed as aids. No calculators, smartwatches, phones, computing devices, or resources such as Wolfram Alpha are allowed. No problems on the exam require the use of a calculator.
6. Figures are not necessarily not drawn to scale.
7. Before beginning the exam, you will ask yourself to record certain information on the answer form if you chose to obtain an AMC 10 Answer Sheet from `https://www.maa.org/math-competitions/amc-10-12/`.
8. When you give yourself the signal, begin working on the problems. You will have **75 minutes** to complete the exam.
9. When you finish the exam, *sign your name* in the space provided at the top of the Answer Sheet should you choose to obtain one from `https://www.maa.org/math-competitions/amc-10-12/`.

The Committee on the Christmas Math Competitions reserves the right to disqualify scores from an individual if it determines that the required security procedures were not followed.

All students will be invited to take the 3rd annual Christmas Invitational Math Examination (CIME) on Friday, January 3, 2020 and Friday, February 7, 2020. More details about the CIME are in the back of this test booklet.

1. There is a positive integer n such that $n\%$ of n is 4. What is $(n^2)\%$ of n?

(A) 10 **(B)** 16 **(C)** 20 **(D)** 80 **(E)** 1600

2. David and Marta are happily married and have three children. In this family of five, the average number of siblings of a member of this family is 4. What is the average of the number of siblings David and Marta have? (Note that David and Marta have siblings who are not part of this family of five.)

(A) 5 **(B)** $\frac{11}{2}$ **(C)** 6 **(D)** $\frac{13}{2}$ **(E)** 7

3. Let n be the smallest positive integer such that the units digit of $2019^n - n$ is a 0. What is the sum of the digits of n^2?

(A) 1 **(B)** 4 **(C)** 7 **(D)** 9 **(E)** 10

4. A group of k children are playing with a 52-card deck. The deck is split so that every child receives a different number of cards, and each child receives at least one card. What is the maximum possible value of k?

(A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

5. The graph of the equation

$$\frac{x+2}{y-7} = \frac{x-5}{y+3}$$

is a line except for two points (a, b) and (c, d). What is $a + b + c + d$?

(A) 3 **(B)** 7 **(C)** 11 **(D)** 13 **(E)** 17

6. What is the sum of the digits of the greatest positive integer n such that $8n + 9$ is divisible by $2n - 5$?

(A) 3 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 15

7. A three-digit positive integer $\underline{abc}$ is *wavy* if $a < b > c$. How many three-digit positive integers are wavy?

(A) 204 **(B)** 240 **(C)** 256 **(D)** 284 **(E)** 285

8. Kayla is flipping 9 fair coins, of which six are silver and three are gold. What is the probability that the number of silver coins that land on heads and the number of gold coins that land on heads are either both even or both odd?

(A) $\frac{1}{3}$ **(B)** $\frac{4}{9}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{5}{9}$ **(E)** $\frac{2}{3}$

9. Josh wants to buy trading cards for 30 cents. He has a bag of pennies with radius 1, nickels with radius 3, dimes with radius 5, and quarters with radius 8. Using these coins, $A\pi$ is the maximum total area of the coins that can be used to purchase the trading cards. What is the sum of the digits of A?

(A) 7 **(B)** 8 **(C)** 10 **(D)** 12 **(E)** 15

10. A positive integer is called a palindrome if it reads the same forwards and backwards. For example, 121 and 777 are palindromes. How many three-digit palindromes are divisible by 6?

(A) 10 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 18

11. In regular hexagon $ABCDEF$, let P_1, P_2, P_3, P_4, P_5, P_6 be the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, $\overline{FA}$, respectively. Lines P_1P_2, P_3P_4, P_5P_6 are the sides of a triangle δ. What is the ratio of the area of δ to the area of $\triangle ACE$?

(A) $\frac{3}{2}$ **(B)** $\frac{27}{16}$ **(C)** 2 **(D)** $\frac{9}{4}$ **(E)** 3

12. Let $(x \heartsuit y) = xy - \frac{x}{y}$ for all nonzero y. What is the product of all values of n satisfying

$$(n \heartsuit 3) = n \cdot (3 \heartsuit n)?$$

(A) -3 **(B)** $-\frac{8}{3}$ **(C)** -1 **(D)** $\frac{8}{9}$ **(E)** 1

13. A line with negative slope passes through $(1,1)$ and has x- and y-intercepts of $(a,0)$ and $(0,b)$ respectively. If $a+b=8$, what is the value of ab?

(A) $4-2\sqrt{2}$ **(B)** 2 **(C)** 4 **(D)** $2+2\sqrt{2}$ **(E)** 8

14. Let $P_1P_2\cdots P_{2020}$ be a regular 2020-gon, and let $P_i = P_{i+2020}$ for all i. For every $i = 1, 2, \ldots, 2020$, draw line P_iP_{i+2}. Into how many regions do these 2020 lines divide the interior of the polygon?

(A) 2020 **(B)** 4040 **(C)** 4041 **(D)** 6060 **(E)** 6061

15. Let $\triangle ABC$ have $AB = 13$, $BC = 14$, and $AC = 15$. Moreover, let O be the circumcenter of $\triangle ABC$ and let D be the foot of the altitude from A to $\overline{BC}$. What is the area of $\triangle AOD$?

(A) 12 **(B)** 14 **(C)** 16 **(D)** 18 **(E)** 20

16. Define the *content* of a nonnegative rational number as its denominator when written in simplest form. In other words, n is the content of $\frac{m}{n}$, where m and n are relatively prime positive integers. Suppose p and q are rational numbers such that the content of p is 5040 and the content of q is 3960. How many possible values are there for the content of $p+q$?

(A) 1 **(B)** 6 **(C)** 10 **(D)** 18 **(E)** 24

17. Oliver selects two nonnegative integers a and b less than 1000 uniformly and at random, and attempts to compute their sum. However, he completely forgets to carry over; so, for instance, to add 13 and 19, he writes down the units digit, 2, discards the carryover, and writes the sum of the tens digits, 2 (thus his answer is 22).

Suppose that the positive difference between his answer and the correct answer is D. What is the expected value of D?

(A) 450 **(B)** 495 **(C)** 499 **(D)** $\frac{999}{2}$ **(E)** 500

18. Let ABC be a triangle with area 1, and let M, N, P be the midpoints of $\overline{BC}$, $\overline{CA}$, $\overline{AB}$, respectively. A point X is chosen uniformly and at random in the interior of $\triangle ABC$. What is the expected value of the area of the convex hull of $\{M, N, P, X\}$?

(The *convex hull* of a finite set S in the plane is the smallest convex polygon containing all the elements of S.)

(A) $\frac{5}{16}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{11}{32}$ **(D)** $\frac{3}{8}$ **(E)** $\frac{1}{2}$

19. Let S be a subset of positive integers with the following properties:

(i) the numbers 1, 2, 3, 4, 5, 6 are elements of S; and

(ii) for all distinct elements a and b in S, ab is also in S.

Let K be the sum of the reciprocals of all the elements of S. What is the minimum possible value of K?

(A) $\frac{91}{30}$ **(B)** $\frac{13}{4}$ **(C)** $\frac{53}{15}$ **(D)** $\frac{37}{10}$ **(E)** $\frac{15}{4}$

20. For positive reals n, an n-minute call costs $n^2 - 3n + 4.41$ dollars. If the minimum cost of a series of phone calls that total to 17 minutes is expressed as $\frac{p}{q}$, where p and q are relatively prime positive integers, what is the sum of digits of p?

(A) 3 **(B)** 13 **(C)** 15 **(D)** 16 **(E)** 19

21. A deck of 8 distinct cards are laid in a stack. In each *shuffling*, a card can be chosen from the stack, removed, and placed at the top of the stack. How many stacks of the cards are there for which a minimum of 4 shufflings are needed to produce this stack from the original stack?

(A) 224 **(B)** 336 **(C)** 630 **(D)** 1344 **(E)** 1680

22. How many ordered pairs of real numbers (a, b) satisfy the equations

$$\lceil a^4 \rceil - 2\lfloor a^2 \rfloor b + \lceil b^2 \rceil^2 = 0$$
$$\lceil a^2 \rceil + 2a\lfloor b^2 \rfloor + \lceil b \rceil^2 = 0?$$

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 8

23. Parallelogram $ABCD$ with $\angle ABC$ acute has side lengths $AB = 17$ and $BC = 20$. Let M and N be the midpoints of $\overline{AD}$ and $\overline{BC}$, respectively. The circumcircle of $\triangle AND$ intersects AB at a point $X \neq A$, and the circumcircle of $\triangle XBN$ intersects segment BM at a point $Y \neq B$. If $CY = 16$, then what is the area of parallelogram $ABCD$?

(A) 280 **(B)** 296 **(C)** 320 **(D)** 336 **(E)** 340

24. Let N be the number of integer sequences $a_1, a_2, \ldots, a_{10}$ such that for every pair (i, j), where $1 \leq i < j \leq 10$,

$$|a_i + a_{i+1} + \cdots + a_j| \leq 2.$$

What is the remainder when N is divided by 100?

(A) 0 **(B)** 25 **(C)** 47 **(D)** 49 **(E)** 99

25. Let N be the number of ordered sextuples of positive integers (a, b, c, d, e, f) satisfying

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} + \frac{1}{f} = 1.$$

What is the remainder when N is divided by 5?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4