2020 CMC 12A Solutions Document

Christmas Math Competitions

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1. **Answer (C):** The function $n \mapsto 2^n$ is strictly increasing, so it is injective. Thus we must have $2^n = n^2$. This holds for the integers 2, 4. To see that no other integers work, note that n must be a power of 2; otherwise for some odd prime p, $\nu_p(2^n) = 0$ but $\nu_p(n^2) \neq 0$.

   However if $n = 2^k$, then $2^n = n^2 = 2^{2k}$, so again $n = 2k$. This means $k = 2^{k-1}$, which only occurs twice. The solutions are $n = 2$ and $n = 4$, and the answer is 2.

2. **Answer (C):** The sum of all six weights is $5+5+10+10+10+20 = 60$, so on the left side of the press, the weights must sum to 30. If we choose the 20-pound weight, we can either choose a single 10-pound weight or two 5-pound weights, giving 2 ways.

   Otherwise, our choices for the left side are $10+10+10$ and $10+10+5+5$. Thus a total of 4 ways are possible.

3. **Answer (B):** Write

   $$f(2019) = 2019^2 + 2 \cdot 2019 + 1 = (2019+1)^2 = 2020^2.$$

   We know 2020 factors as $2^2 \cdot 5 \cdot 101$, so the requested sum is $2 + 5 + 101 = 108$.

4. **Answer (C):** The resulting set is $S' = \{2, 3, 10, 15, 26, 35, n \pm 1\}$. If the median of S is not n, then $n \pm 1$ must be the median of S, and also one of $\{1, 4, 9, 16, 25, 36\}$. This two assertions contradict each other, so n must be the median of S.

   This forces $9 < n < 16$. If the median of S' is also $n \pm 1$, then $n \neq n \pm 1$, so the median of S' is either 9 or 16. It follows that $n \in \{10, 15\}$, both of which can be easily seen to work. The answer is 2.

5. **Answer (A):** The sequence begins $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, ... This pattern leads to the conjecture that $a_n = \frac{n}{n+1}$. Checking the base case $n = 0$ and observing that for $n \geq 1$,

   $$a_n = \frac{1}{2 - a_{n-1}} = \frac{1}{2 - \frac{n-1}{n}} = \frac{1}{\frac{n+1}{n}} = \frac{n}{n+1}$$

   confirms the conjecture. Therefore

   $$a_1 a_2 \cdots a_{2019} = \frac{1}{2} \cdot \frac{2}{3} \cdots \frac{2019}{2020} = \frac{1}{2020},$$

   the answer.

6. **Answer (D):** Tasty's 6-sided die is weighted so that the output is 1 (mod 5) with probability $\frac{1}{3}$, and any other residue with probability $\frac{1}{6}$ each. This implies that to maximize Stacy's chances of winning, her die should maximize the probability of rolling a number 4 (mod 5). It is clear that a 9-sided die is the best of the answer choices.

7. **Answer (C):** The first statement must be true. Otherwise, the fourth statement is false, and the first statement is true, contradiction. As a corollary, the fourth statement must be false, and we are left to determine the states of the middle two statements.

   If the second statement is true, then the third statement is true. If the second statement is false, then the third statement can be both true or false. The number of subsets is then $1 + 2 = 3$.

8. **Answer (A):** Write $z = a + bi$. The left-hand expression rewrites to

$$(a + bi - 1)(a - bi + 1) = a^2 - (1 - bi)^2 = \left(a^2 + b^2 - 1\right) + 2bi.$$

   Then $a^2 + b^2 = 2020$, and we want to maximize $2b$. By the trivial inequality, $a^2 \geq 0$, so $b^2 = 2020 - a^2 \leq 2020$. At equality, $2b \leq 2\sqrt{2020} < 90$, so the largest possible n is 89.

9. **Answer (D):** To pick a starting point for our search, we consider the least b_4 so that $\frac{b_4}{4^2} > \frac{b_6}{6^2}$; that is,

$$\frac{9}{4} > \frac{b_6}{b_4} = \frac{24 - b_4}{b_4} \implies b_4 > \frac{96}{13} > 7.$$

   As such, we check $(b_4, b_6) = (8, 16)$. This yields

$$\frac{8}{4^2} > \frac{12}{5^2} > \frac{16}{6^2},$$

   so $b_5 = 12$.

   Note: To see that no other $b_4 > 8$ work, check that in these cases,

$$\frac{b_4}{4^2} - \frac{b_6}{6^2} \geq \frac{9}{16} - \frac{15}{36} = \frac{7}{48} > \frac{2}{25},$$

   so we conclude by the Pidgeonhole Principle that there are at least two $b_5/5^2$, with $b_5 \in \mathbb{Z}$, obeying

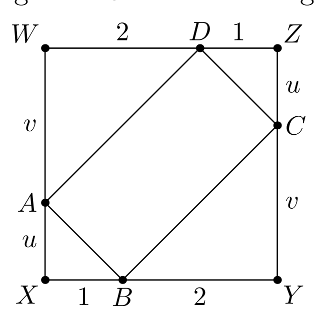
$$\frac{b_4}{4^2} > \frac{b_5}{5^2} > \frac{b_6}{6^2}.$$

10. **Answer (C):** Let the common value be x. Write

$$x = \sqrt{n + x} = \frac{1000}{n + x}.$$

   If $y = n + x$, then $\sqrt{y} = 1000/y$, so $y^{3/2} = 1000$ and $y = 100$. This tells us the common value is $x = 10$, so $n = 100 - 10 = 90$.

11. **Answer (B):** Without loss of generality A, B, C, D have x-coordinates 1, 2, 4, 3 respectively. Inscribe the rectangle $ABCD$ in a rectangle $WXYZ$ as shown.



Let $u = AX = CZ$ and $v = AW = CY$. By hypothesis, $BX = DZ = 1$ and $BY = DW = 2$. Since $\angle XAB = 90^\circ - \angle ABX = \angle YBC$, by AA, $\triangle AXB \sim \triangle BYC$, so

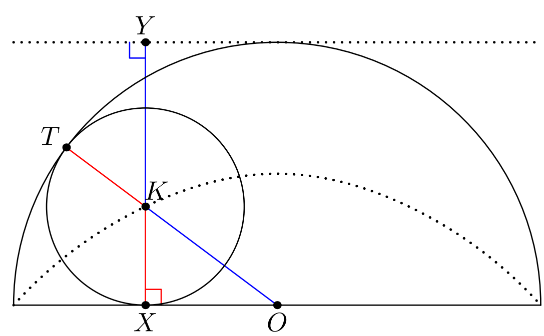
$$\frac{u}{1} = \frac{2}{v} \implies uv = 2.$$

The area of $ABCD$ is given by $3(u+v)$, the areas of $\triangle AXB$ and $\triangle CZD$ are both given by $u/2$, and the areas of $\triangle BYC$ and $\triangle DWA$ are both given by v. As such,

$$\text{Area}(WXYZ) = 3(u+v) - 2 \cdot \frac{u}{2} - 2 \cdot v = u + 2v \geq 2\sqrt{2uv} = 4,$$

where the inequality is given by the AM-GM inequality. Take the equality case of the inequality, $u = 2v \iff (u,v) = (1,2)$, to achieve the lower bound of 4.

12. **Answer (D):** Consider the diagram below. Let O be the center of Γ, K the center of a circle $\omega \in S$, T the touch point of Γ and ω, X the touch point of ω and $\overline{AB}$, and Y the point on $\overline{KX}$ such that the tangent ℓ from Y to Γ is parallel to $\overline{AB}$ (and thus ℓ is fixed).



Seeing that $XY = OT$ is the radius of Γ and $KX = KT$ is the radius of ω, we deduce $KY = KO$. Thus K lies on a parabola with focus O and directrix ℓ.

13. **Answer (D):** The answer is 30. To see that 30 is achievable, consider the set defined by $S = \{1, 2, 3, \ldots, 30\}$. Only the first 10 primes divide some element of S.

    To see that no S with $|S| = 31$ has primality 10, note that the 11 primes 2, 3, ..., 31 all divide at least one number in S, as any consecutive stretch of n numbers contains some number divisible by n.

14. **Answer (C):** We use the estimate

$$m = \left\lfloor \log(9!^{9!}) \right\rfloor = \lfloor 9! \log(9!) \rfloor \approx 9! \log(9!).$$

Since $9! \log(9!) > 300{,}000$, this bound has almost zero effect on the answer. Similarly, $n \approx 10! \log(10!)$, so

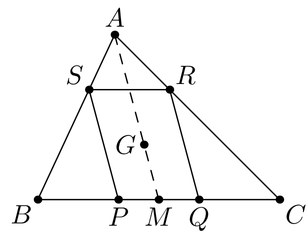
$$\frac{n}{m} = \frac{10! \log(10!)}{9! \log(9!)} = 10 \log_{9!}(10!) = 10\left(1 + \frac{1}{\log(9!)}\right).$$

The estimate $5 < \log(9!) < 6$ gives

$$11.5\overline{6} < \frac{n}{m} < 12.$$

Since $x \mapsto 1/x$ is convex, $\frac{1}{11.5}$ is closer to $\frac{1}{12}$ than $\frac{1}{11}$, so the answer must be $\frac{1}{12}$.

15. **Answer (D):** Let G be the centroid and M the midpoint of $\overline{BC}$. In general, the area of $PQRS$ is $\frac{4}{9}$ of the area of $\triangle ABC$.



Let $d(X,\ell)$ be the distance from X to ℓ and $d(\ell_1,\ell_2)$ be the distance between parallel lines ℓ_1, ℓ_2. Since $d(A,\overline{BC}) = 3d(G,\overline{BC})$ and G is the center of $PQRS$, we have $d(\overline{RS},\overline{BC}) = \frac{2}{3}d(A,\overline{BC})$. This implies $AR = \frac{1}{3}AC$ and $AS = \frac{1}{2}AB$.

Now $\overline{AG}$ bisects $\overline{RS}$ by homothety, so the midpoint of $\overline{PQ}$ is M. Then $PQ = RS = \frac{1}{3}BC$, so $BP = PQ = QC$. Finally

$$\frac{[ASR]}{[ABC]} = \frac{1}{9}, \quad \frac{[BPS]}{[ABC]} = \frac{2}{9}, \quad \frac{[CQR]}{[ABC]} = \frac{2}{9},$$

and combining these results yields the desired conclusion. Alternatively the base of $PQRS$ is $\frac{1}{3}$ that of $\triangle ABC$, and its height is $\frac{2}{3}$ that of $\triangle ABC$, and the result readily follows as well.

From the given numbers, $[ABC] = 756$, so the answer is 336.

16. **Answer (C):** Note that a and b end up in the numerator and denominator respectively, but c, d, e can go anywhere. That is, for all $x, y, z \in \{1, -1\}$, the expression can attain $ab^{-1}c^x d^y e^z$. Optimally the exponents of 2, 3, 5, 7, 9 in N are all ± 1, whereas all other exponents are 0. Say that there are p exponents of 1 and q exponents of -1.

We can choose a in p ways and b in q ways. Next, c, d, e may be any of the 6 permutations of the remaining primes, and x, y, z follow accordingly. Then the number of permutations is given by $6pq \le 6 \cdot 2 \cdot 3 = 36$, the answer.

17. **Answer (D):** Let $x = \log a$ and $y = \log(b/a)$, so that x, $x+y$, $\log\gamma$, $x+3y$ form a geometric progression. It follows that

$$\frac{x+3y}{x+y} = \frac{(x+y)^2}{x^2} \implies y^2(3x+y) = 0 \implies y = -3x.$$

Then $x \log C = (x-3x)^2$, so $\log\gamma = 4\log a$ and $\gamma = a^4 = 4$.

18. **Answer (C):** Observe that

$$\tan\theta + \tan\left(\frac{\pi}{2} - \theta\right) = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{2}{\sin(2\theta)},$$

so the sum can be written as

$$2\left[\frac{1}{\sin(\pi/12)} + \frac{1}{\sin(\pi/4)} + \frac{1}{\sin(5\pi/12)}\right] = \sqrt{96} + \sqrt{8},$$

and the requested sum is $96 + 8 = 104$.

19. **Answer (B):** Let $x = \log_2 a$ and $y = \log_2 b$, so that

$$\frac{xy}{2} + 4^x = 135 \quad \text{and} \quad \frac{xy}{2} + 4^y = 263.$$

Seeing as 135 and 263 are both 7 away from a power of 2, we conjecture $xy/2 = 7$, or $xy = 14$. Indeed, $4^x = 128 \implies x = 7/2$ and $4^y = 256 \implies y = 4$, which work.

Thus $ab = 2^{15/2} = 128\sqrt{2}$, and the requested sum is $128 + 2 = 130$.

20. **Answer (C):** Note that G must be the center of homothety between $ABCD$ and $P_8P_7P_5P_3$, so G lies on $\angle P_1AP_8$. It follows that

$$\tan\angle BAG = \frac{P_1P_8}{AP_1} = \frac{1}{\varphi^3} = \sqrt{5}-2,$$

where $\varphi = \frac{1+\sqrt{5}}{2}$.

21. **Answer (A):** Note that $216,000 = 60^3 = 2^6 \cdot 3^3 \cdot 5^3$. Let $p \in \{2,3,5\}$ be prime, and let $x = \nu_p(a)$, $y = \nu_p(b)$, $z = \nu_p(c)$. Note that it is equivalent that x, y, z has no unique maximum[1].

Say that e is the exponent of p. There are $e+1$ ways to choose $x = y = z$. Furthermore, if $x = y > z$, then $x = y \in \{1, \ldots, e\}$, and z can be chosen in $x = y$ ways. Thus the number of ways to determine x, y, z is

$$(e+1) + 3\sum_{i=1}^{e} i = e+1+\frac{3e(e+1)}{2} = \frac{(3e+2)(e+1)}{2}.$$

This equals 70 when $e = 6$ and 22 when $e = 3$, so the answer is $70 \cdot 22^2 = 33,880$.

22. **Answer (C):** Note that A, B, C are the D-, E-, F-excenters of $\triangle DEF$. Henceforth let $a = EF$, $b = FD$, $c = DE$, $s = \frac{1}{2}(a+b+c)$, and let K be the area. Hence

$$\frac{K}{s-a} = 2, \quad \frac{K}{s-b} = 3, \quad \frac{K}{s-c} = 6.$$

From this,

$$1 = \frac{1}{2}+\frac{1}{3}+\frac{1}{6} = \frac{s-a}{K}+\frac{s-b}{K}+\frac{s-c}{K} = \frac{s}{K},$$

so $s = K$.

By Heron's formula, $K^2 = s(s-a)(s-b)(s-c)$, so $K = (s-a)(s-b)(s-c)$ and

$$K^2 = \frac{K}{s-a}\cdot\frac{K}{s-b}\cdot\frac{K}{s-c} = 36,$$

whence $K = 6$.

23. **Answer (A):**

First solution, by Allen Baranov (unedited) Draw the angle bisector AD. Let $CD = m$ and $BD = n$. Then,

$$\frac{b}{m} = \frac{c}{n} = \frac{b+c}{m+n} = \frac{b+c}{a} = 2\cos C.$$

Therefore, $\frac{b}{m} = 2\cos C$ and $b = 2m\cos C$. Applying Law of Cosines on triangle ADC for AD shows $AD = m$. Therefore, $\angle DAC = \angle C$.

Let $\angle DAC = a$. Then $\angle B + \angle A + \angle C = 30 + 2a + a = 180°$ and $2a = 100°$.

OR

Second solution, by Kaiwen Li (unedited) Note that $\frac{b+c}{2\cos C} = a \implies a^2 = b(b+c)$. Thus, if we construct a point D on ray BA such that $AD = AC$, it would follow that circle (ADC) is tangent to $\overline{BC}$. Trivial angle chasing yields $\angle A = 100°$.

OR

[1] that is, the largest and second-largest among x, y, z are equal

Third solution, by Eric Shen Alternatively we can just force the good ol' sine ratio trick. Draw in the angle bisector $\overline{AD}$, and let $\angle A = 2\theta$. Note that $\angle ADC = 30^\circ + \theta$ and $\angle ACB = 150^\circ - 2\theta$. Then by the law of sines,

$$\frac{\sin(30^\circ + \theta)}{\sin\theta} = \frac{AB}{BD} = 2\cos C = \frac{\sin 2C}{\sin C} = \frac{\sin(300^\circ - 4\theta)}{\sin(150^\circ - 2\theta)}.$$

From here it is clear $\theta = 50^\circ$, and $\angle A = 100^\circ$.

24. **Answer (C):**

Solution by Justin Lee For each four-digit number $\overline{abcd}$, we first pick the two pairs of numbers that have an equal sum. There are 3 ways to do this.

This common sum could range from 1 to 18; if the sum equals 1, then there is 1 way to pick the pair of numbers that includes a, and there are 2 ways for the other pair. Repeating this until 18, we see there are a total of

$$3(1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + 9\cdot 10 + 9\cdot 9 + 8\cdot 8 + \cdots + 1\cdot 1) = 1845$$

numbers. However, we overcounted some numbers — in partiular, $\overline{aabb}$, $\overline{abab}$, $\overline{abba}$. As there are two ways to form an equal sum for each of these numbers when $a \neq b$, we must subtract $3\cdot 81$. Now when the number is of the form $\overline{aaaa}$ we must subtract two ways, giving $1845 - 3\cdot 81 - 2\cdot 9 = 1584$.

25. **Answer (B):**

Solution, by Federico Clerici (unedited) Let $0 \le m, h < 12$ be the positions of the minute and hour hands, respectively. Note that we are talking about the position on the clock, and not about the number of minutes: hence, for example, 9:30 corresponds to $(h, m) = (9.5, 6)$.

Clearly, using this notation, $\{h\} = \frac{m}{12} \implies m = 12\{h\}$. Hence, all the times when we can't exactly read the clock are the pairs $(h, m) = (x, y) = (y, x)$, that is to say, all the points of intersections of lines $x = 12\{y\}$ and $y = 12\{x\}$ with $x \neq y$. We can see that the solutions lie on the set of parallel lines $y = x + \frac{12k}{13}$, where $|k| = 1, 2, \ldots, 11$.

Since we require $|x - y| \le 3 \implies \frac{12|k|}{13} \le 3 \implies |k| \le 3$. Hence, the maximum error is $|x - y| = \frac{12\cdot 3}{13}$ hours, and in minutes this corresponds to $60\cdot\frac{12\cdot 3}{13} \approx 166.15$, and the answer is 166.