2020

CMC 12B

DO NOT OPEN UNTIL SUNDAY, January 26, 2020

Christmas Math Competitions

Questions and comments about problems and solutions for this exam should be sent by PM to:

AOPS12142015, **FedeX333X**, and **TheUltimate123**.

The 3rd Annual CIME will be held on Friday, January 3, 2020, with the alternate on Friday, February 7, 2020. It is a 15-question, 3-hour, integer-answer exam. You will be invited to participate regardless of your score on this competition. All students will be invited to take the 3rd Annual Christmas American Math Olympiad (CAMO) or the Christmas Junior Math Olympiad (CJMO) on Friday, January 10, 2020.

A complete listing of our previous publications may be found at our web site:

`http://cmc.ericshen.net/`

****Administration On An Earlier Date Will Literally Be Impossible****

1. All the information needed to administer this exam is contained in the non-existent CMC 10/12 Teacher's Manual. PLEASE READ THE MANUAL EVERY DAY BEFORE JANUARY 26, 2020.
2. YOU must not verify on the CMC 10/12 COMPETITION CERTIFICATION FORM (found on `maa.org/amc` under "AMC 12B") that you followed all rules associated with the administration of the exam.
3. If you chose to obtain an AMC 10 Answer Sheet from the MAA's website, it must be returned to yourself the day after the competition. Ship with inappropriate postage without using a tracking method. FedeX333X or UPS is strongly recommended.
4. The publication, asexual reproduction, sexual reproduction, or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the definite (but not indefinite) integrity of the results. Dissemination via phone, email, raven, or digital media of any type during this period is a violation of the competition rules.

The **Christmas Math Competitions**
is made possible by the contributions of the following problem-writers and test-solvers:

David Altizio, Allen Baranov, Luke Choi,
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Preston Fu, Justin Lee, Kyle Lee, Kaiwen Li,
Sean Li, Eric Shen, Anthony Wang, Andrew Wen,
Nathan Xiong, *and* Joseph Zhang

MAC CMC
Christmas Math Competitions

Christmas Math Competitions
3rd Annual

CMC 12B

Christmas Math Contest 12B
Sunday, January 26, 2020

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU HAVE STARTED YOUR TIMER.
2. This is a 25-question multiple-choice exam. Each question is followed by answers marked A, B, C, D, and E. Only one of these is correct.
3. Mark your answer to each problem however you want. If you would like to create a more realistic test experience, then you may obtain an AMC 10 Answer Sheet from `https://www.maa.org/math-competitions/amc-10-12/` and mark you answer to each problem on the AMC 10 Answer Sheet with a #2 pencil. To simulate the real test, check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer sheet will be graded in a real test. For the CMC, **you must submit your answers using the Submission Form found at** `http://cmc.ericshen.net/CMC-2020/`**. Only answers submitted to the Submission Form will be scored.**
4. Scoring: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. Only scratch paper, graph paper, rulers, compasses, protractors, and erasers are allowed as aids. No calculators, smartwatches, phones, computing devices, or resources such as Wolfram Alpha are allowed. No problems on the exam require the use of a calculator.
6. Figures are not necessarily not drawn to scale.
7. Before beginning the exam, you will ask yourself to record certain information on the answer form if you chose to obtain an AMC 10 Answer Sheet from `https://www.maa.org/math-competitions/amc-10-12/`.
8. When you give yourself the signal, begin working on the problems. You will have **75 minutes** to complete the exam.
9. When you finish the exam, *sign your name* in the space provided at the top of the Answer Sheet should you choose to obtain one from `https://www.maa.org/math-competitions/amc-10-12/`.

The Committee on the Christmas Math Competitions reserves the right to disqualify scores from an individual if it determines that the required security procedures were not followed.

All students will be invited to take the 3rd annual Christmas Invitational Math Examination (CIME) on Friday, January 3, 2020 and Friday, February 7, 2020. More details about the CIME are in the back of this test booklet.

1. There is a positive integer n such that $n\%$ of n is 4. What is $(n^2)\%$ of n?

(A) 10 **(B)** 16 **(C)** 20 **(D)** 80 **(E)** 1600

2. A group of k children are playing with a 52-card deck. The deck is split so that every child receives a different number of cards, and each child receives at least one card. What is the maximum possible value of k?

(A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

3. What is the sum of the digits of the greatest positive integer n such that $8n + 9$ is divisible by $2n - 5$?

(A) 3 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 15

4. Josh wants to buy trading cards for 30 cents. He has a bag of pennies with radius 1, nickels with radius 3, dimes with radius 5, and quarters with radius 8. Using these coins, $A\pi$ is the maximum total area of the coins that can be used to purchase the trading cards. What is the sum of the digits of A?

(A) 7 **(B)** 8 **(C)** 10 **(D)** 12 **(E)** 15

5. A positive integer is called a palindrome if it reads the same forwards and backwards. For example, 121 and 777 are palindromes. How many three-digit palindromes are divisible by 6?

(A) 10 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 18

6. In regular hexagon $ABCDEF$, let P_1, P_2, P_3, P_4, P_5, P_6 be the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, $\overline{FA}$, respectively. Lines P_1P_2, P_3P_4, P_5P_6 are the sides of a triangle δ. What is the ratio of the area of δ to the area of $\triangle ACE$?

(A) $\frac{3}{2}$ **(B)** $\frac{27}{16}$ **(C)** 2 **(D)** $\frac{9}{4}$ **(E)** 3

7. Suppose $(x \heartsuit y) = xy - \frac{x}{y}$ for all nonzero y. What is the product of all values of n satisfying

$$(n \heartsuit 3) = n \cdot (3 \heartsuit n)?$$

(A) -3 **(B)** $-\frac{8}{3}$ **(C)** -1 **(D)** $\frac{8}{9}$ **(E)** 1

8. What is the sum of the squares of the real parts of the roots of $x^2 - 2\sqrt{5}x + (5 - 8i)$?

(A) 1 **(B)** $2\sqrt{5}$ **(C)** 10 **(D)** 18 **(E)** 19

9. Let $\triangle ABC$ have $AB = 13$, $BC = 14$, and $AC = 15$. Moreover, let O be the circumcenter of $\triangle ABC$ and let D be the foot of the altitude from A to $\overline{BC}$. What is the area of $\triangle AOD$?

(A) 12 **(B)** 14 **(C)** 16 **(D)** 18 **(E)** 20

10. Isosceles right triangle ABC with right angle at B has area 1. Let M be the midpoint of $\overline{AB}$. A line ℓ passing through M and perpendicular to AC intersects segment AC at X and the extension of line BC at Y. What is the area of triangle $\triangle CXY$?

(A) 1 **(B)** $\dfrac{3\sqrt{2}}{4}$ **(C)** $\dfrac{3+\sqrt{2}}{4}$ **(D)** $\dfrac{9}{8}$ **(E)** $\dfrac{11+6\sqrt{2}}{16}$

11. Let $a_1, a_2, \ldots, a_n$ be a sequence of positive integers such that no two terms are relatively prime. Suppose the product of all the terms of the sequence is $2^1 \cdot 3^2 \cdot 5^3 \cdot 7^4 \cdot 11^5$. What is the maximum possible value of n?

(A) 6 **(B)** 11 **(C)** 15 **(D)** 16 **(E)** 21

12. Define the *content* of a nonnegative rational number as its denominator when written in simplest form. In other words, n is the content of $\frac{m}{n}$, where m and n are relatively prime positive integers. Suppose p and q are rational numbers such that the content of p is 5040 and the content of q is 3960. How many possible values are there for the content of $p + q$?

(A) 1 **(B)** 6 **(C)** 10 **(D)** 18 **(E)** 24

13. Oliver selects two nonnegative integers a and b less than 1000 uniformly and at random, and attempts to compute their sum. However, he completely forgets to carry over; so, for instance, to add 13 and 19, he writes down the units digit, 2, discards the carryover, and writes the sum of the tens digits, 2 (thus his answer is 22).

Suppose that the positive difference between his answer and the correct answer is D. What is the expected value of D?

(A) 450 **(B)** 495 **(C)** 499 **(D)** $\dfrac{999}{2}$ **(E)** 500

14. Let ABC be a triangle with area 1, and let M, N, P be the midpoints of $\overline{BC}$, $\overline{CA}$, $\overline{AB}$, respectively. A point X is chosen uniformly and at random in the interior of $\triangle ABC$. What is the expected value of the area of the convex hull of $\{M, N, P, X\}$?

(The *convex hull* of a finite set S in the plane is the smallest convex polygon containing all the elements of S.)

(A) $\frac{5}{16}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{11}{32}$ **(D)** $\frac{3}{8}$ **(E)** $\frac{1}{2}$

15. Let $r_1 < r_2 < r_3$ be the roots of the cubic $x^3 - 3x - 1$ and let $s_1 < s_2 < s_3$ be the roots of the cubic $x^3 - 3x + 1$. What is the value of $|(r_1 + s_1)(r_2 + s_2)(r_3 + s_3)|$?

(A) 0 **(B)** 1 **(C)** 3 **(D)** 6 **(E)** 9

16. Trapezoid $ABCD$ with $\overline{AB} \parallel \overline{CD}$ is inscribed in a circle ω. The tangent to ω at A intersects line CD at P, and the tangent to ω at C intersects line AB at Q. Lines AP and CQ intersect at T. Suppose that $DA = AB = BC = 5$ and $CD = 10$. What is $TP \cdot TQ$?

(A) 50 **(B)** 60 **(C)** 75 **(D)** 100 **(E)** 150

17. For a n-minute call, where n is a positive real number, a phone line charges $n^2 - 3n + 4.41$ dollars. If the minimum cost of a series of phone calls that total to 17 minutes is expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers, what is the sum of digits of a?

(A) 3 **(B)** 13 **(C)** 15 **(D)** 16 **(E)** 19

18. A deck of 8 distinct cards are laid in a stack. In each *shuffling*, a card can be chosen from the stack, removed, and placed at the top of the stack. How many stacks of the cards are there for which a minimum of 4 shufflings are needed to produce this stack from the original stack?

(A) 224 **(B)** 336 **(C)** 630 **(D)** 1344 **(E)** 1680

19. Let $ABCD$ be a convex quadrilateral such that $AB = 4$, $BC = 4$, $CD = 3$, $DA = 7$. There exists a unique point P inside quadrilateral $ABCD$ such that the areas of $\triangle PAB$, $\triangle PBC$, $\triangle PCD$, $\triangle PDA$ are all numerically equal. What is the value of $PA^2 + PB^2 + PC^2 + PD^2$?

(A) 30 **(B)** 35 **(C)** 40 **(D)** 45 **(E)** 50

20. How many ordered pairs of real numbers (a, b) satisfy the equations

$$\lceil a^4 \rceil - 2\lfloor a^2 \rfloor b + \lceil b^2 \rceil^2 = 0$$
$$\lceil a^2 \rceil + 2a\lfloor b^2 \rfloor + \lceil b \rceil^2 = 0?$$

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 8

21. Let ABC be an isosceles triangle with side lengths $AB = AC = 8$, and let ω and I be the incircle and incenter of $\triangle ABC$, respectively. Suppose ω is tangent to $\overline{BC}$, $\overline{AC}$, $\overline{AB}$ at D, E, F, respectively. Segment AI intersects ω at X. Given that lines XE, FI, BC intersect at a common point, then the length of $\overline{BC}$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. What is the value of $m + n$?

(A) 17 **(B)** 18 **(C)** 19 **(D)** 20 **(E)** 21

22. What is the remainder when

$$\left(1^{2020} + 2^{2020} + \cdots + 2019^{2020}\right)^{2020}$$

is divided by 2020?

(A) 0 **(B)** 1 **(C)** 404 **(D)** 1616 **(E)** 2019

23. In acute $\triangle ABC$, let D and E be the feet of the altitudes from A to $\overline{CB}$ and C to $\overline{AB}$, respectively. Suppose that $AB = 4$, $CB = 3$, and $DE = 2$. The sum of all possible values of the length of $\overline{AC}$ can be written in the form $\frac{\sqrt{p}+q}{r}$, where p is not divisible by the square of any prime. What is the value of $p + q + r$?

(A) 72 **(B)** 74 **(C)** 75 **(D)** 78 **(E)** 92

24. Let N be the number of integer sequences $a_1, a_2, \ldots, a_{10}$ such that for every pair (i, j), where $1 \le i < j \le 10$,

$$|a_i + a_{i+1} + \cdots + a_j| \le 2.$$

What is the remainder when N is divided by 100?

(A) 0 **(B)** 25 **(C)** 47 **(D)** 49 **(E)** 99

25. How many monic polynomials $P(x)$ of degree 12, whose coefficients are integers between 0 and 12, inclusive, have the property that for all integers x, either $P(x) + 1$ or $P(x) - 1$ is divisible by 13?

(A) 792 **(B)** 924 **(C)** 1001 **(D)** 1716 **(E)** 3003