CMC ARML Competition 2019

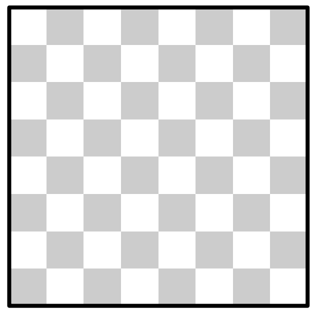
AOPS12142015
Blast_S1
Cosmicgenius
djmathman
eisirrational
FedeX333X
illogical_21
jeffisepic
jj_ca888
mfang92
novus677
pretzel
Th3Numb3rThr33
TheUltimate123
WannabeCharmander

May 13–31, 2019

1 Individual Problems

Problem 1. In triangle ABC, $\angle A = 60^\circ$ and $BC = 12$. Given that the perimeter of $\triangle ABC$ is 30, compute the area of $\triangle ABC$.

Problem 2. Consider the set of rectangles formed by the unit squares of an 8×8 checkerboard. Compute the number of these rectangles contain exactly four black squares.



Problem 3. Compute $\dfrac{3^{\log_{2/3} 3}}{2^{\log_{2/3} 2}}$.

Problem 4. The numbers $a_1, a_2, a_3, \dots$ form an arithmetic sequence. Suppose that there exist unique positive integers $p < q < 63$ such that $a_p = p^2$, $a_q = q^2$, and $a_{63} = 2019$. Compute $p + q$.

Problem 5. Each second, two randomly chosen digits of 123 are swapped. Compute the expected number of seconds until the number 321 is formed.

Problem 6. Let $\triangle ABC$ be an isosceles triangle with $AB = AC = 1$, and let E and F be the feet of the altitudes from B and C to sides $\overline{AC}$ and $\overline{AB}$, respectively. If line EF is tangent to the incircle of $\triangle ABC$, compute the perimeter of $\triangle ABC$.

Problem 7. In acute triangle ABC, $AB = 2$ and $AC = 3$. Let D be the foot of the angle bisector of $\angle BAC$. The circle with diameter AD intersects $\overline{AB}$ and $\overline{AC}$ at E and F, respectively. Given that the area of $\triangle DEF$ is one sixth of the area of $\triangle ABC$, compute BC^2.

Problem 8. There exist unique positive integers $1 < a < b$ satisfying

$$a^2 + b^2 = 2^{20} + 1.$$

Compute $a + b$.

Problem 9. Let a and b be real numbers such that the polynomial

$$P(x) = x^3 - ax^2 + 11ax - b$$

has three distinct real roots forming a geometric progression. Compute b.

Problem 10. Isabel and Robert play a game in which Isabel chooses a list of 6 real numbers and Robert devises 6 questions, each of the form "What is the sum of the x^{th} and y^{th} number in your list?" where $1 \le x < y \le 6$. Without regard to order, compute the number of ways can Robert choose his questions so that he can determine Isabel's numbers.

2 Answers to Individual Problems

Answer 1. $15\sqrt{3}$

Answer 2. 120

Answer 3. $\frac{1}{6}$ (or $0.1\overline{6}$)

Answer 4. 37

Answer 5. 5

Answer 6. $1+\sqrt{5}$ (or $\sqrt{5}+1$)

Answer 7. $13-2\sqrt{11}$ (or $-2\sqrt{11}+13$)

Answer 8. 1385

Answer 9. 1331

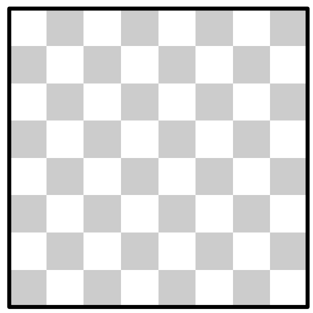
Answer 10. 2530

3 Solutions to Individual Problems

Problem 1. In triangle ABC, $\angle A = 60^\circ$ and $BC = 12$. Given that the perimeter of $\triangle ABC$ is 30, compute the area of $\triangle ABC$.

Solution 1. Set $x = AB$ and $y = AC$. By the Law of Cosines on $\triangle ABC$, $x^2+y^2-xy = 144$, and by the perimeter condition, $x + y = 30 - BC = 18$. Hence, $x^2 + y^2 + 2xy = 324$, and subtracting gives $3xy = 180 \implies xy = 60$. Thus the answer is $[ABC] = \frac{\sqrt{3}}{4}xy = \mathbf{15\sqrt{3}}$.

Problem 2. Consider the set of rectangles formed by the unit squares of an 8×8 checkerboard. Compute the number of these rectangles contain exactly four black squares.



Solution 2. It is not hard to check that the area of these rectangles are either 7, 8, or 9. We take cases:

- Case 1: Area 7. The rectangle must have dimensions 1×7 and contain 4 black squares. Each row and column has exactly one such rectangle, so there are 16 rectangles in this case.
- Case 2: Area 8. The rectangle has dimensions of either 1×8 or 2×4. Each row and column contains a 1×8 rectangle, for a total of 16. We can choose the top left corner of a 2×4 rectangle with long side horizontal in $5 \cdot 7 = 35$ ways, and similarly a 4×2 rectangle in 35 ways, so there are $16 + 2 \cdot 35 = 86$ rectangles in this case.
- Case 3: Area 9. The rectangle has dimensions 3×3, and its corners must be white. The top-left corner can be chosen in $3 \cdot 6 = 18$ ways, and thus it is the number of rectangles in this case.

The answer is $16 + 86 + 18 = \mathbf{120}$.

Problem 3. Compute $\dfrac{3^{\log_{2/3} 3}}{2^{\log_{2/3} 2}}$.

Solution 3. Let $p = \log_{2/3} 2$ and $q = \log_{2/3} 3$, so that $p - q = 1$; we desire $\frac{3^q}{2^p}$. However, we are given that

$$\frac{2^p}{3^p} = 2 \implies 2^p = 2 \cdot 3^p \implies \frac{3^q}{2^p} = \frac{3^q}{2 \cdot 3^p} = \frac{1}{2 \cdot 3^{q-p}} = \mathbf{\frac{1}{6}},$$

the answer.

Problem 4. The numbers $a_1, a_2, a_3, \ldots$ form an arithmetic sequence. Suppose that there exist unique positive integers $p < q < 63$ such that $a_p = p^2$, $a_q = q^2$, and $a_{63} = 2019$. Compute $p + q$.

Solution 4. Let a_0 be the term before a_1 and d the common difference such that $a_0 + pd = p^2$, $a_0 + qd = q^2$, and $a_0 + 63d = 63^2$. Subtracting the first two equations yields $(q-p)d = q^2 - p^2$, whence $d = p + q$. It is now easy to check that $a_0 = -pq$. The third equation yields

$$63(p+q) - pq = 2019 \implies (p-63)(q-63) = 63^2 - 2019 = 1950 = 2 \cdot 3 \cdot 5^2 \cdot 13.$$

Now, $-63 < p, q < 0$, so $\{p-63, q-63\} = \{-50, -39\}$, whence $p = 13$, $q = 24$, and $p + q = \mathbf{37}$.

Problem 5. Each second, two randomly chosen digits of 123 are swapped. Compute the expected number of seconds until the number 321 is formed.

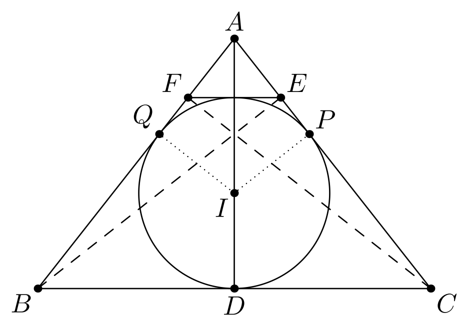
Solution 5. Let E_1 and E_2 be the expected number of seconds when one and two flips away from 321, respectively. From E_1, there is a $\frac{1}{3}$ chance of reaching 321 and a $\frac{2}{3}$ chance of reaching E_2. From E_2 every step reaches E_1, whence

$$\begin{cases} E_2 = E_1 + 1, \\ E_1 = \frac{2}{3}E_2 + 1, \end{cases}$$

from which it follows that $E_1 = \mathbf{5}$.

Problem 6. Let $\triangle ABC$ be an isosceles triangle with $AB = AC = 1$, and let E and F be the feet of the altitudes from B and C to sides $\overline{AC}$ and $\overline{AB}$, respectively. If line EF is tangent to the incircle of $\triangle ABC$, compute the perimeter of $\triangle ABC$.

Solution 6. Let $x = \sin \frac{A}{2}$, and denote by h_A and r the length of the A-altitude and the inradius, respectively. Furthermore let I denote the incenter and $\triangle DPQ$ the contact triangle of $\triangle ABC$.



Notice that since $\triangle AFE \sim \triangle ABC$ with scale factor $\cos A : 1$, we have that $1 - \cos A = \frac{2r}{h_A}$. Clearly $BD = DC = x$, so the semiperimeter of $\triangle ABC$ is $1 + x$, and $AP = AQ = 1 - x$. Then, $r = AP \tan \frac{A}{2} = (1-x)\tan\frac{A}{2}$. Since $AD = \cos \frac{A}{2}$,

$$2x^2 = 1 - \cos A = \frac{2(1-x)\tan\frac{A}{2}}{\cos\frac{A}{2}} = \frac{2x(1-x)}{\cos^2\frac{A}{2}} = \frac{2x(1-x)}{1-x^2}.$$

Since $x \neq 0$, we can simplify this to $x = \frac{1}{1+x}$, so $x^2 + x - 1 = 0$. As $-1 \le x \le 1$, $x = \frac{\sqrt{5}-1}{2}$, and the perimeter of $\triangle ABC$ is $2x + 2 = \mathbf{1 + \sqrt{5}}$, the answer.

Problem 7. In acute triangle ABC, $AB = 2$ and $AC = 3$. Let D be the foot of the angle bisector of $\angle BAC$. The circle with diameter AD intersects $\overline{AB}$ and $\overline{AC}$ at E and F, respectively. Given that the area of $\triangle DEF$ is one sixth of the area of $\triangle ABC$, compute BC^2.

Solution 7. Note that $\angle AED = \angle AFD$ and $\triangle AED \cong \triangle AFD$. Let $DE = DF = x$, so that

$$x^2 \sin A = 2[DEF] = \frac{1}{3}[ABC] = \frac{1}{6} \cdot 2 \cdot 3 \sin A = \sin A \implies x = 1.$$

It is immediate that $[ABC] = [ABD] + [ACD] = \frac{5}{2}$, so

$$\frac{5}{2} = [ABC] = \frac{1}{2} \cdot 2 \cdot 3 \sin A \implies \sin A = \frac{5}{6},$$

whence $\cos A = \frac{\sqrt{11}}{6}$ and

$$BC^2 = 2^2 + 3^2 - 2 \cdot 2 \cdot 3 \cos A = \mathbf{13 - 2\sqrt{11}},$$

the answer.

Problem 8. There exist unique positive integers $1 < a < b$ satisfying

$$a^2 + b^2 = 2^{20} + 1.$$

Compute $a + b$.

Solution 8. By the Brahmagupta–Fibonacci identity,

$$\left(a^2 + b^2\right)\left(c^2 + d^2\right) = (ac + bd)^2 + (ad - bc)^2 = (ac - bd)^2 + (ad + bc)^2.$$

Check that $2^4 + 1 \mid 2^{20} + 1 = \left(2^{10}\right)^2 + 1$, so we have that $ac + bd = 2^{10}$, $ad - bc = 1$, and $c^2 + d^2 = 17$.

- Case 1: $c = 1$ and $d = 4$. We have that $a + 4b = 2^{10}$ and $4a - b = 1$, whence $17a = 2^{10} + 4$, which is absurd.
- Case 2: $c = 4$ and $d = 1$. We have that $4a + b = 2^{10}$ and $a - 4b = 1$, whence $17b = 2^{10} - 4$, which yields $a = 241$ and $b = 60$.

The answer is then $(ac - bd) + (ad + bc) = (964 - 60) + (241 + 240) = 904 + 481 = \mathbf{1385}$.

Problem 9. Let a and b be real numbers such that the polynomial

$$P(x) = x^3 - ax^2 + 11ax - b$$

has three distinct real roots forming a geometric progression. Compute b.

Solution 9. Let k be the middle root and $r > 1$ be the common ratio; we can see that the roots are $kr, k, \frac{k}{r}$. Note that by Vieta's the sum of the roots is

$$kr + k + \frac{k}{r} = k\left(r + 1 + \frac{1}{r}\right) = a$$

and the sum of the pairwise products of the roots is

$$kr \cdot k + kr \cdot \frac{k}{r} + k \cdot \frac{k}{r} = k^2\left(r + 1 + \frac{1}{r}\right) = 11a,$$

from which

$$\frac{k^2\left(r + 1 + \frac{1}{r}\right)}{k\left(r + 1 + \frac{1}{r}\right)} = \frac{11a}{a} \Rightarrow k = 11.$$

The answer is $b = kr \cdot k \cdot \frac{k}{4} = k^3 = 11^3 = \mathbf{1331}$.

Problem 10. Isabel and Robert play a game in which Isabel chooses a list of 6 real numbers and Robert devises 6 questions, each of the form "What is the sum of the x^{th} and y^{th} number in your list?" where $1 \leq x < y \leq 6$. Without regard to order, compute the number of ways can Robert choose his questions so that he can determine Isabel's numbers.

Solution 10. We can consider the the game as a graph with 6 vertices, representing Isabel's numbers, and 6 edges, representing Robert's questions. Note that we can't have a cycle with even length since then the corresponding system of equations wouldn't be linearly independent. For example, the system of equations $x + y = c_1$, $y + z = c_2$, $z + w = c_3$, $w + x = c_4$ is not linearly independent, since the sum of the 1^{st} and 3^{rd} equations is equal to the sum of the 2^{nd} and 4^{th}; hence one equation is wasted, yet 6 independent equations are required to find 6 variables. Since the largest tree in a graph with 6 vertices consists of 5 edges, a cycle must exist. We distinguish three cases, depending on the length of the largest cycle:

- Case 1: we have a 5-cycle. There are 6 ways to choose which number is left out of the 5-cycle, 12 ways to order the elements of the 5-cycle, and 5 ways to connect the number left out to the cycle, for a total of 360 possibilities.
- Case 2: there are two 3-cycles. As the cycles are disjoint, there are $\frac{\binom{6}{3}}{2} = 10$ such possibilities.
- Case 3: there is only one 3-cycle. We choose the cycle $\mathcal{C}$ in $\binom{6}{3} = 20$ ways, and then choose the paths to connect the other three vertices X, Y, Z to $\mathcal{C}$, where each of these points must have exactly 1 path. If $X \to \mathcal{C}$, $Y \to \mathcal{C}$ and $Z \to \mathcal{C}$, there are $3^3 = 27$ ways; if $X \to Y \to \mathcal{C}$ and $Z \to \mathcal{C}$ (and permutations), there are $6 \cdot 3^2 = 54$ ways; if $X \to Z$, $Y \to Z$ and $Z \to \mathcal{C}$ (and permutations), there are $3 \cdot 3 = 9$ ways; finally, if $X \to Y \to Z \to \mathcal{C}$ (and permutations), there are $6 \cdot 3 = 18$ ways, for a total of $20 \cdot (27 + 54 + 9 + 18) = 2160$ possibilites.

Thus, Robert has a total of $360 + 10 + 2160 =$ **2530** ways to ask Isabel his questions.

4 Relay Problems

Relay 1-1. Compute

$$\cfrac{\cfrac{\cfrac{\cfrac{1}{1+1}}{1+2+1}}{1+2+2+1}}{1+2+3+2+1} \Bigg/ \ldots$$

$$\frac{\dfrac{\dfrac{\dfrac{1}{1+1}}{1+2+1}}{1+2+2+1}}{\dfrac{1+2+3+2+1}{\dfrac{1+2+2+1}{\dfrac{1+2+1}{\dfrac{1+1}{1}}}}}.$$

Relay 1-2. Let $T = \text{TNYWR}$ and $K = \frac{1}{T}$. Suppose $\{a_n\}$ is a sequence of real numbers satisfying $a_n a_{n+2} + a_{n+1} = 1$ for all $n \geq 1$. Given that $a_1 = 2$ and $a_2 = 3$, compute $a_1 + a_2 + \ldots + a_K$.

Relay 1-3. Let $T = \text{TNYWR}$. Rectangle $ABCD$ has perimeter T and satisfies $AB = 2BC$. Suppose that the circle centered at A passing through D intersects the circle centered at C passing through D at a point D and P. Compute the length of BP.

Relay 2-1. Jack is standing at 0 on the number line. After every second, if he is at position n, he moves to either $n+3$, $n+1$, $n-1$, or $n-3$, each with an equal probability. The probability that Jack is back at 0 after 4 seconds is $\frac{p}{q}$, where p and q are relatively prime positive integers. Compute p.

Relay 2-2. Let $T = \text{TNYWR}$. Let f be the function defined by $f(x) = x^4 - x^3 + x^2 - x + 1$, and let $g(x) = f(x+T)$. Compute the product of roots of $g(x)$.

Relay 2-3. Let $T = \text{TNYWR}$ and let S be the sum of the digits of T. Suppose f is a function satisfying $f(2^n) = n^2$ and $f(3n) = f(2n) + 1$ for all nonnegative integers n. Compute $f(3^S)$.

5 Relay Answers

Answer 1-1. $\frac{1}{144}$ (or $0.0069\overline{4}$)

Answer 1-2. 126

Answer 1-3. $\frac{63\sqrt{5}}{5}$ $\left(\text{or } \frac{63}{\sqrt{5}}\right)$

Answer 2-1. 11

Answer 2-2. 13421

Answer 2-3. 132

6 Relay Solutions

Relay 1-1. Compute

$$\frac{\frac{\frac{\frac{1}{1+1}}{1+2+1}}{1+2+2+1}}{1+2+3+2+1}\Bigg/\frac{1+2+2+1}{\frac{1+2+1}{\frac{1+1}{1}}}.$$

Solution 1-1. Starting to simplify the fraction from the top, we see that the numerator is

$$\frac{1}{1+1}\cdot\frac{1}{1+2+1}\cdot\frac{1}{1+2+2+1}=\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{6}=\frac{1}{48};$$

on the other hand, starting to simplify the denominator from the bottom,

$$\frac{1+1}{1}=2,\quad \frac{1+2+1}{2}=2,\quad \frac{1+2+2+1}{2}=3,\quad \frac{1+2+3+2+1}{3}=3.$$

Thus the answer is $\frac{1}{48}\cdot\frac{1}{3}=\mathbf{\frac{1}{144}}$.

Relay 1-2. Let $T=$ TNYWR and $K=\frac{1}{T}$. Suppose $\{a_n\}$ is a sequence of real numbers satisfying $a_na_{n+2}+a_{n+1}=1$ for all $n\geq 1$. Given that $a_1=2$ and $a_2=3$, compute $a_1+a_2+\ldots+a_K$.

Solution 1-2. Listing out the first terms of the sequence, we can see that $\{a_n\}$ has a period of 5, since $a_3=-1$, $a_4=\frac{2}{3}$, $a_5=-\frac{1}{3}$, $a_6=2$ and $a_7=3$. We are given that $K=\frac{1}{T}=144$, thus

$$\sum_{n=1}^{144}a_n=29\sum_{n=1}^{5}a_n-a_5=29\cdot\frac{13}{3}+\frac{1}{3}=\mathbf{126}.$$

Relay 1-3. Let $T=$ TNYWR. Rectangle $ABCD$ has perimeter T and satisfies $AB=2BC$. Suppose that the circle centered at A passing through D intersects the circle centered at C passing through D at a point D and P. Compute the length of BP.

Solution 1-3. Since P is the reflection of D across AC, it also lies on the semicircle with diameter AC. Since $AB+BC=\frac{T}{2}$ and $AB=2BC$, we have that $AB=\frac{T}{3}$ and $BC=\frac{T}{6}$. Applying Ptolemy's Theorem to $APBC$,

$$\frac{T}{3}\cdot\frac{T}{3}=\frac{T}{6}\cdot\frac{T}{6}+BP\cdot\sqrt{\left(\frac{T}{3}\right)^2+\left(\frac{T}{6}\right)^2}\implies BP=\frac{T}{2\sqrt{5}}.$$

Plugging in $T=126$, we see that the answer is $BP=\mathbf{\frac{63}{\sqrt{5}}}$.

Relay 2-1. Jack is standing at 0 on the number line. After every second, if he is at position n, he moves to either $n+3$, $n+1$, $n-1$, or $n-3$, each with an equal probability. The probability that Jack is back at 0 after 4 seconds is $\frac{p}{q}$, where p and q are relatively prime positive integers. Compute p.

Solution 2-1. Denote the possible moves by $+3,+1,-1,-3$. We distinguish three cases:

- Case 1: $+3,+1,-1,-3$. Since the moves are all distinct, there are $4!=24$ such possibilities.
- Case 2: $x,x,-x,-x$. We have 2 possible choices for x (either 1 or 3) and $\binom{4}{2}=6$ possible orderings, for a total of $2\cdot 6=12$ possibilities.

- Case 3: $\pm1, \pm1, \pm1, \mp3$. We have 2 possible choices for the $\pm$ sign and 4 possible orderings, for a total of $2 \cdot 4 = 8$ possibilities.

Since the total number of possible moves without any restrictions is $4^4 = 256$, the desired probability is $\frac{44}{256} = \frac{11}{64}$, and $m = \mathbf{11}$.

Relay 2-2. Let $T =$ TNYWR. Let f be the function defined by $f(x) = x^4 - x^3 + x^2 - x + 1$, and let $g(x) = f(x+T)$. Compute the product of roots of $g(x)$.

Solution 2-2. Since $f(x)$ is a polynomial with degree 4, $g(x) = f(x+T)$ is a polynomial with degree 4 too. Hence, by Vieta's, the product of the roots of $g(x)$ is equal to the constant term $g(0) = f(T) = T^4 - T^3 + T^2 - T + 1$. Plugging in $T = 11$, we see that the answer is **13421**.

Relay 2-3. Let $T =$ TNYWR and let S be the sum of the digits of T. Suppose f is a function satisfying $f(2^n) = n^2$ and $f(3n) = f(2n) + 1$ for all nonnegative integers n. Compute $f(3^S)$.

Solution 2-3. We can prove by induction that $f(3^n 2^{S-n}) = S^2 + n$. Indeed, if $n = 0$, then $f(3^0 2^S) = f(2^S) = S^2 + 0$ by the first condition. Suppose now that our claim holds true for a certain value of $n > 1$. Then, using the second condition,

$$f\left(3^{n+1}2^{S-n-1}\right) = f\left(3 \cdot [3^n 2^{S-n-1}]\right) = f\left(2 \cdot [3^n 2^{S-n-1}]\right) + 1 = f\left(3^n 2^{S-n}\right) + 1 = S^2 + n + 1,$$

proving our claim. Since $f(3^S) = f(3^S 2^{S-S}) = S^2 + S$, plugging in $S = 11$ gives $11^2 + 11 = \mathbf{132}$, the answer.

7 Tiebreaker Problem

Problem 1. Compute the smallest prime factor of $1(1+2(2+3(3+\cdots+9(9+10\cdot 10)\cdots)))$.

8 Tiebreaker Answer

Answer 1. 13

9 Tiebreaker Solution

Problem 1. Compute the smallest prime factor of $1(1 + 2(2 + 3(3 + \cdots + 9(9 + 10 \cdot 10) \cdots)))$.

Solution 1. Check that the expression is equal to

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + 10 \cdot 10! = 11! - 1,$$

whence $p > 11$. If $p = 13$ then by Wilson's theorem,

$$11! \equiv \frac{12!}{12} \equiv \frac{-1}{-1} \equiv 1 \pmod{13},$$

so the answer is **13**.