MAC CMC

Christmas Mathematics Competitions

Christmas Mathematics Competitions

2^{nd} Annual

CIME I

Solutions Pamphlet

Christmas Invitational Mathematics Examination I Solutions Pamphlet

Friday, December 28, 2018

This Solutions Pamphlet gives at least one solution for each problem on this year's CIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

Correspondence about the problems and solutions for this CIME and orders for any of the publications listed below should be PM'd to:

AOPS12142015, eisirrational, FedeX333X, and TheUltimate123.

The problems and solutions for this CIME were prepared by the CMC's Committee on the CIME under the direction of:

Kyle Lee
CMC Chair
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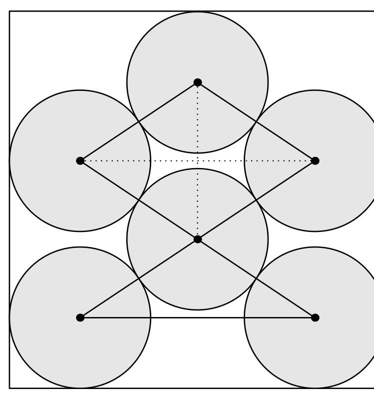
1. (Answer: 585)

   Notice that for an integer N to have four digits in base 8, $N \geq 512$. However, the first digit then must be 1, as if the first digit is at least 2, then $N \geq 1024$, which is undesirable. Then, $N = \underline{1}\,\underline{\text{d}}\,\underline{\text{d}}\,\underline{1}_8$, for some digit d. Then, $N = 72\text{d} + 513$, and testing values of d with $0 \leq \text{d} \leq 7$ gives that the largest such that N is a base-10 palindrome is $\text{d} = 1$,and $N = 585$.

2. (Answer: 027)



   Connect the centers of the four circles not tangent to the bottom edge. We then have a rhombus with side length 2. Let the diagonals of this rhombus be d_1 and d_2, with d_1 horizontal and d_2 vertical. By the Pythagorean Theorem,

$$\left(\frac{d_1}{2}\right)^2 + \left(\frac{d_2}{2}\right)^2 = 4 \implies d_1^2 + d_2^2 = 16.$$

   Now, consider the isosceles triangle formed by the centers of the three bottommost squares. Since the height of this isosceles triangle is $\frac{d_2}{2}$, the side length of the square can be expressed as $\frac{3d_2}{2} + 2$. However, the length of the base of this triangle is d_1, so the length of the square can be expressed as $d_1 + 2$. Then, $d_1 = \frac{3d_2}{2}$, so

$$16 = d_1^2 + d_2^2 = \frac{13d_2^2}{4} \implies d_2 = \frac{8}{\sqrt{13}},$$

   and the side length of the square is $\frac{12}{\sqrt{13}} + 2$. The desired sum is $12 + 13 + 2 = 27$.

3. (Answer: 429)

   We first count the number of positive integers in the sequence that are less than 2000. All 333 multiples of 3 that are between 1000 and 2000 are in the sequence. Now, we need to count the number of integers in the sequence that are less than 1000; in other words, the number of 3-digit strings of digits (not necessarily nonzero) $\underline{a}\,\underline{b}\,\underline{c}$ with $1 \in \{a, b, c\}$ and $a + b + c \equiv 0 \pmod 3$. We have two cases:

- *Case 1:* Besides the digit 1, the two other digits are congruent to 0 and 2 modulo 3 in some order. There are 4 ways to choose the 0 (mod 3) digit, 3 to choose the 2 (mod 3) digit, and $3! = 6$ to arrange the digits, showing that there are $4 \cdot 3 \cdot 6 = 72$ numbers in this case.
- *Case 2:* The numbers are all 1 (mod 3). If all of the digits are 1, we only have 1 number. If two of the digits are 1, we have 2 ways to choose the other digit and 3 ways to arrange the digits, giving $2 \cdot 3 = 6$ numbers. If only one of the digits is a 1, and the other two numbers are the same, we have 2 ways to choose the other digit and 3 ways to arrange them, giving 6 numbers. Finally, if only one of the digits is a 1, and the other two numbers are distinct, there is $\binom{2}{2} = 1$ way to choose the digits and 6 ways to arrange them. Then, the number of integers in this case is $1 + 6 + 6 + 6 = 19$.

There are then $333+72+19 = 424$ numbers in the sequence less than 2000. Clearly, 2001, 2010, 2013, and 2016 are the only integers in the sequence that are greater than 2000 and before 2019, so 2019 is the 429$^{\text{th}}$ term of the sequence.

4. (Answer: 021)

We claim that $a_{2n-1} = 3 \cdot 3^{n-2}$ and $a_{2n} = 5 \cdot 3^{n-2}$ for all positive integers $n > 1$. We prove this by induction. Clearly the base case is true. Then,

$$\begin{aligned} a_{2n+1} = a_{2(n+1)-1} &= a_{2n} + a_{2n-1} + \gcd(a_{2n}, a_{2n-1}) \\ &= 3 \cdot 3^{n-2} + 5 \cdot 3^{n-2} + 3^{n-2} = 9 \cdot 3^{n-2} = 3 \cdot 3^{(n+1)-2}, \end{aligned}$$

and similarly $a_{2(n+1)} = 5 \cdot 3^{(n+1)-2}$, hence our claim is true.

Now, our summation is only

$$\begin{aligned} \sum_{k=2}^{\infty} \frac{1}{a_{k+1} - a_k} &= \frac{1}{a_3 - a_2} + \sum_{k=3}^{\infty} \frac{1}{a_{k+1} - a_k} \\ &= \frac{1}{3-1} + \sum_{n=2}^{\infty} \frac{1}{a_{2n} - a_{2n-1}} + \frac{1}{a_{2n+1} - a_{2n}} \\ &= \frac{1}{2} + \sum_{n=2}^{\infty} \frac{1}{5 \cdot 3^{n-2} - 3 \cdot 3^{n-2}} + \frac{1}{9 \cdot 3^{n-2} - 5 \cdot 3^{n-2}} \\ &= \frac{1}{2} + \sum_{n=2}^{\infty} \frac{1}{2 \cdot 3^{n-2}} + \frac{1}{4 \cdot 3^{n-2}} \\ &= \frac{1}{2} + \frac{3}{4} \sum_{n=2}^{\infty} \frac{1}{3^{n-2}} \\ &= \frac{1}{2} + \frac{3}{4} \cdot \frac{3}{2} = \frac{13}{8}, \end{aligned}$$

and the desired sum is $13 + 8 = 21$.

5. (Answer: 187)

Let $r = \log_9 a, s = \log_9 b, t = \log_9 c$. First, factor

$$p(x) = x^3 - kx^2 + kx - 1 = (x-1)(x^2 - (k-1)x + 1).$$

Then, WLOG let $t = 1$, so that $c = 9$ and $x^2 - (k-1)x + 1$ has roots r and s. We then have

$$9^r + 9^s = a + b = 84,\ r + s = k - 1,\ \text{and } rs = 1.$$

Since $rs = 1$, $s = \frac{1}{r}$, so $9^r + 9^{1/r} = 84$. Clearly $r = 2, \frac{1}{2}$ satisfy this. Because for $x > 1$, the function $9^x + 9^{1/x}$ is strictly increasing, these are our only possible r. Then,

$$k = r + s + 1 = 2 + \frac{1}{2} + 1 = \frac{7}{2} \implies n = 9^{7/2} = 2187,$$

and the requested remainder is 187.

6. (Answer: 032)

The key claim is that $d_3 < \sqrt{n} < d_4$. We prove this by contradiction. If $d_3 < d_4 \leq \sqrt{n}$, then

$$d_3^2 + d_4^2 \leq 2n < 2n + 1,$$

which is a violation of the given conditions. Analogously, if $\sqrt{n} \leq d_3 < d_4$, then

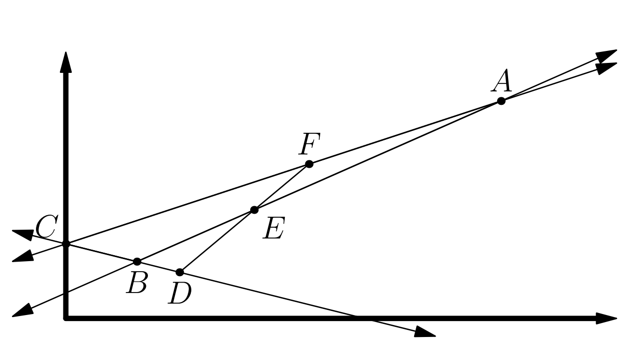
$$d_3^2 + d_4^2 \geq (\sqrt{n})^2 + (\sqrt{n} + 1)^2 = 2n + 2\sqrt{n} + 1 > 2n + 1,$$

which too is impossible. Hence, n has 6 divisors; more specifically, $d_3 d_4 = n$, and so

$$d_3^2 + d_4^2 = 2d_3 d_4 + 1 \implies (d_4 - d_3)^2 = 1 \implies d_4 = d_3 + 1.$$

Since d_3, d_4 and consecutive integers, one of them must be even, and so $2 \mid n$. It follows that $(d_1, d_2, d_3, d_4, d_5, d_6) = (1, 2, d_3, d_3 + 1, \frac{n}{2}, n)$. Since all divisors of d_3 are divisors of n, d_3 must either be prime or only divisible by $1, 2, d_3$, whence $d_3 = 4$. Since $d_3 \nmid d_4$, d_4 must either be prime or equal to 4. Since one of d_3, d_4 must be even, they cannot both be prime. Hence, one of them is 4, and $d_3 = 3, 4$. Both cases are valid, and so our two values of n are $3 \cdot 4 = 12$ and $4 \cdot 5 = 20$. The answer is $12 + 20 = 32$.

7. (Answer: 245)

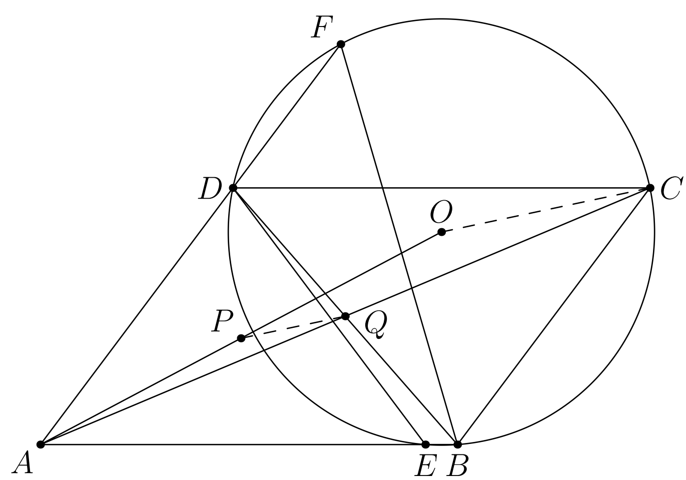


Draw the position-time graphs of Albert, Bob, Carrie, and Douglas, representing the lines by a, b, c, d, respectively. Suppose that $A = b \cap c, B = c \cap a, C = a \cap b, D = d \cap a, E = d \cap c, F = d \cap b$. Then, by Menelaus's Theorem on $\triangle ABC$ with traversal $\overline{DEF}$,

$$1 = \frac{AF}{FC} \cdot \frac{CD}{DB} \cdot \frac{BE}{EA} = \frac{m-77}{77} \cdot \frac{32}{12} \cdot \frac{33}{m-53} = \frac{8(m-77)}{7(m-53)}.$$

It follows that $8m - 8 \cdot 77 = 7m - 7 \cdot 53$, so $m = 8 \cdot 77 - 7 \cdot 53 = 7(88-53) = 245$, the answer.

8. (Answer: 084)



$\measuredangle$ denotes a directed angle. Note that since

$$\measuredangle BFA = \measuredangle BFD = \measuredangle BCD = \measuredangle DAB = \measuredangle FAB,$$

$BA = BF$. Similarly, since $\measuredangle DEA = \measuredangle EAD$, $DA = DE$. As a result, $BF \cdot DE = AB \cdot AD$. Let R denote the circumradius of $\triangle BCD$. By symmetry, R is also the circumradius of $\triangle ABD$. However, since we are given that the length of the altitude from A to $\overline{BD}$ has length 7,

$$R = \frac{AB \cdot AD \cdot BD}{4 \cdot \frac{1}{2} \cdot BD \cdot 7} \implies AB \cdot AD = 14R.$$

However, since $ABCD$ is a parallelogram, Q is the midpoint of $\overline{AC}$, so

$$R = OC = 2PQ = 6 \implies BF \cdot DE = 14 \cdot 6 = 84,$$

the desired answer.

9. (Answer: 033)

Assume that $i < j$. The second condition then becomes $a_i < b_{ij} < a_j$. The key claim is that for such a sequence to satisfy the conditions, the second condition being satisfied by all $j = i+1$ is a sufficient condition. This is because if $i < j < k$ and the condition is satisfied for (i, j) and (j, k), since $\{b_i\}$ is strictly increasing,

$$a_i < b_{ij} < b_{ik} < b_{jk} < a_k,$$

proving our claim.

Now, we only have that $a_1 < b_{1\cdot 2} < a_2 < b_{2\cdot 3} < a_3 < \cdots < a_{19}$. Then, for all $1 < i < 19$, $b_{(i-1)i} < a_i < b_{i(i+1)}$. Since a_i is the only integer in the sequence between $b_{(i-1)i}$ and $b_{i(i+1)}$, the number of possible values for a_i is $(2i-1)+1 = 2i$. Then, we seek

$$\nu_2\left(\prod_{i=2}^{18} 2i\right) = \nu_2\left(2^{17} \cdot 18!\right) = 17 + 16 = 33,$$

the answer.

10. (Answer: 009)

Let $N = x^2 + y^2 + z^2$. Notice that

$$(y-z)^2 = x^2 + y^2 + z^2 - (x^2 + 2yz) = N - 3.$$

Similarly, $(x-z)^2 = N - 4$ and $(y-x)^2 = N - 5$. Notice that since $y - z = (x-z) + (y-x)$, we have

$$\begin{aligned}
\sqrt{N-3} &= \sqrt{N-4} + \sqrt{N-5} \\
N - 3 &= 2N - 9 + 2\sqrt{(N-4)(N-5)} \\
6 - N &= 2\sqrt{(N-4)(N-5)} \\
36 - 12N + N^2 &= 4N^2 - 36N + 80 \\
0 &= 3N^2 - 24N + 44 \\
N &= 4 + \frac{2}{\sqrt{3}},
\end{aligned}$$

and the requested sum is $4 + 2 + 3 = 9$.

11. (Answer: 065)

Let $y = ax$ and $z = by$, where $a, b \geq 2$, so that every multiplicative positive integer can be expressed as $N = x + ax + abx = x(1 + a + ab)$. Note that if n is multiplicative, for all k, kn is multiplicative by taking $x \mapsto kx$ and keeping a, b constant.

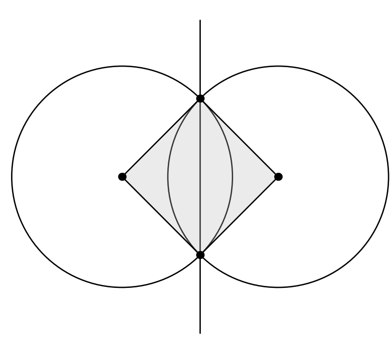
If $x = 1$ and $a = 2$, then $N = 1 + 2(b+1)$, which is a valid expression for every odd number $N \geq 7$. Furthermore, every prime $p \geq 7$ is multiplicative, and so is every number divisible by a prime at least 7. Therefore, we only need consider $N = 2^r 3^s 5^t$. Notice that for all k, taking $(x, a, b) = (k, 2, 6)$ gives $N = 15k$, so every multiple of 15 is multiplicative. For the same reason, if $s \geq 2$ or $t \geq 2$, N is divisible by an odd number at least 7, and therefore N is multiplicative. We now only have three cases:

- *Case 1:* $N = 2^r$. Since $x(1+a+ab) \geq 7$, $1, 2, 4$ are not multiplicative. Also, 8 is not multiplicative, because if $8 = x(1+a+ab) = x(1+a(b+1))$, a factor of 8 must be $1+a(b+1) \geq 7$, so $1+a(b+1) = 8$, which is impossible. However, $(x, a, b) = (1, 3, 4)$ gives $N = 16$, so every multiple of 16 is multiplicative.
- *Case 2:* $N = 2^r \cdot 3$. Note that $3, 6, 12, 24$ are not multiplicative as they do not have any factor at least 7 that can be written as $1 + a + ab$. But, $16 \mid 48$, so any multiple of 48 is multiplicative.
- *Case 3:* $N = 2^r \cdot 5$. Since $5 < 7$, 5 is not multiplicative. However, $(x, a, b) = (1, 3, 2)$ gives $N = 10$, so any multiple of 10 is multiplicative.

Finally, the desired sum is $1 + 2 + 3 + 4 + 5 + 6 + 8 + 12 + 24 = 65$.

12. (Answer: 285)

On the xy-plane, $\mathcal{T}$ is simply the union of the graphs of $y = x - 1$ for $x \geq 0$ and $y = -x - 1$ for $x \leq 0$. Notice that the transformation $\mathcal{T} \mapsto \mathcal{T}'$ is analogous to inversion at the origin with radius 1 followed by a 90° clockwise rotation. We ignore the rotation, as rotation is a rigid transformation and preserves area. Now, fixing $(-1, 0), (\pm 1, 0)$, the resulting graph is the union of two circles centered at $(-1, -1)$ and $(1, -1)$, both with radius $\frac{1}{\sqrt{2}}$.



We can then compute that the area of the union is

$$\mathcal{A} = 2 \cdot \frac{3}{4} \left(\frac{1}{\sqrt{2}} \right)^2 \pi + \left(\frac{1}{\sqrt{2}} \right)^2 = \frac{3}{4}\pi + \frac{1}{2}.$$

Then, we seek

$$\lfloor 100\mathcal{A} \rfloor = \lfloor 75\pi + 50 \rfloor = 285,$$

the answer.

13. (Answer: 057)

Obviously $a, b, c \neq 0$. Since $a(a-b) = b(b-c) = c(c-a) = 1$,

$$a - b = \frac{1}{a},\ b - c = \frac{1}{b}, \text{ and } c - a = \frac{1}{c}.$$

Summing gives

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0 \implies ab + bc + ca = 0.$$

Summing the given equations also gives

$$\begin{aligned} 3 &= a^2 + b^2 + c^2 - ab - bc - ca \\ &= (a+b+c)^2 - 3(ab+bc+ca) \\ &= (a+b+c)^2 \\ \implies a+b+c &= \sqrt{3}. \end{aligned}$$

Since $a - b = \frac{1}{a}$, $a^2 = ab + 1$, and so multiplying gives

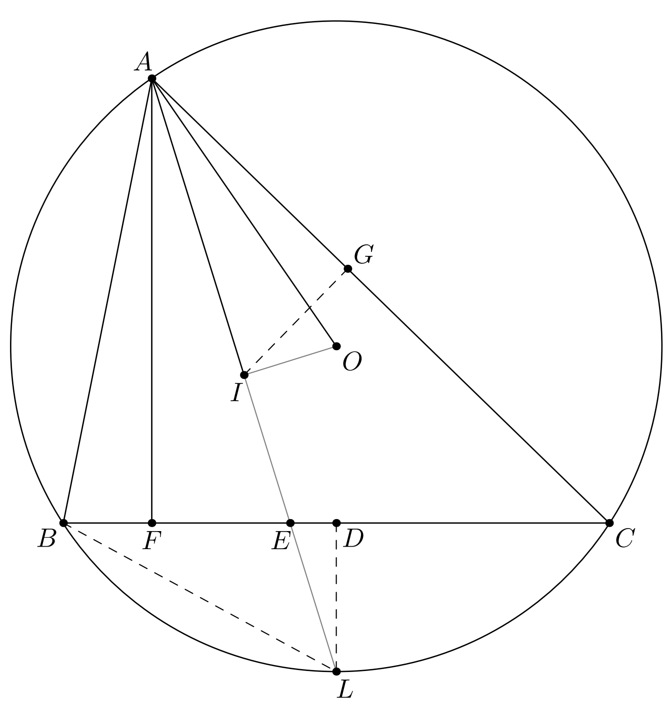
$$\begin{aligned} (abc)^2 &= (ab+1)(bc+1)(ca+1) \\ &= (abc)^2 + abc(a+b+c) + (ab+bc+ca) + 1 \\ \implies abc &= -\frac{1}{\sqrt{3}}. \end{aligned}$$

Since P is monic,

$$\mathrm{P}(x) = x^3 - \sqrt{3}x^2 + \frac{1}{\sqrt{3}} \implies 100|\mathrm{P}(\sqrt{3})| = \frac{100}{\sqrt{3}},$$

and the answer is 57.

14. (Answer: 851)



To begin, define E as the foot of the A-bisector, F as the foot of the altitude from A, and L as the midpoint of $\widehat{BC}$. Then, by the Incenter-Excenter Lemma, $LB = LI = LC = d$ for some d. By Ptolemy's Theorem on $ABLC$,

$$BC \cdot AD = AB \cdot d + AC \cdot d = 2BC \cdot d \implies d = \frac{AD}{2}.$$

It follows that I is the midpoint of $\overline{AD}$

Now, note that $\triangle LBE \sim \triangle LAB$, so

$$LE = \frac{LB^2}{LA} = \frac{AI}{2} \implies AE = \frac{3AI}{2} = 87.$$

Because $\angle OAI = \angle EAF$ and $\angle OIA = \angle EFA$, $\triangle EAF \sim \triangle OAI$ and

$$AF = \frac{AI \cdot AE}{AO} = \frac{87 \cdot 58}{60} = \frac{841}{10},$$

so the requested sum is $841 + 10 = \boxed{851}$.

Alternate Solution. Like above, we see that $\overline{AI} \perp \overline{IO}$. By Euler's Formula for the triangle,

$$60^2 - 58^2 = OI^2 = 60^2 - 2 \cdot 60 \cdot r,$$

whence $r = \frac{58^2}{2 \cdot 60} = \frac{841}{30}$. Then, since $s = \frac{3}{2}a$, $h_a = 3r = \frac{841}{10}$.

15. (Answer: 404)

The key claim is that $\mathcal{F}_k(n)$ is the number of k-tuples with a product of n. We prove this by induction. Obviously, the number of 1-tuples with a fixed product is $\mathcal{F}_1(n) = 1$. Now, for the inductive step, to evaluate the number of k-tuples with a product of n, we can pick the last element and let it be any divisor of n. Then, the product of the rest of the divisors is also a unique divisor of n, so the desired number of k-tuples is the sum of $\mathcal{F}_{k-1}(d)$ for all $d \mid n$, proving our claim.

Factor $864 = 2^5 3^3$ and $648 = 2^3 3^4$. We will evaluate $\mathcal{F}_{2019}(2^a 3^b)$. Note that the distribution of the powers of 2 is independent of the distribution of the powers of 3. By Stars and Bars, the number of ways to distribute the powers of 2 is $\binom{2018+a}{2018}$ and the number of ways to distribute the powers of 3 is $\binom{2018+b}{2018}$, whence

$$\mathcal{F}_{2019}(2^a 3^b) = \binom{2018+a}{2018}\binom{2018+b}{2018},$$

and we have

$$\frac{\mathcal{F}_{2019}(864)}{\mathcal{F}_{2019}(648)} = \frac{\binom{2023}{2018}\binom{2021}{2018}}{\binom{2021}{2018}\binom{2022}{2018}} = \frac{4!(2023 \cdot 2022 \cdot 2021 \cdot 2020 \cdot 2019)}{5!(2022 \cdot 2021 \cdot 2020 \cdot 2019)} = \frac{2023}{5}.$$

Hence, the answer is 404, and we are done.