The 2nd CIME I Solutions Pamphlet will be released after the contest.

CONTACT US – *Correspondence about the problems and solutions for this CIME should be sent by PM to:*

AOPS12142015, eisirrational, FedeX333X, *and* TheUltimate123.

The problems and solutions for this AIME were prepared by the CMC's Committee on the CIME under the direction of:

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2019 CJMO – THE CHRISTMAS MATHEMATICAL JUNIOR OLYMPIAD (CJMO) is a 6-question, 9-hour, essay-type examination. The CJMO will be held from Friday, January 4, 2019 to Friday, January 25, 2018. Your teacher will not have more details on who qualifies for the CJMO in the CMC 10/12 and CIME Teachers' Manuals because we did not make Teachers' Manuals and all students are qualified for the CJMO. The best way to prepare for the CJMO is to study previous years of these exams, which may be found on our website as indicated below.

PUBLICATIONS – For a complete listing of our previous competitions please visit our website at `https://sites.google.com/view/annualcmc/`.

The
MAC Christmas Mathematics Competitions
are supported by the following problem-writers and test-solvers:
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MAC CMC
Christmas Mathematics Competitions

Christmas Mathematics Competitions

2^{nd} Annual

CIME I

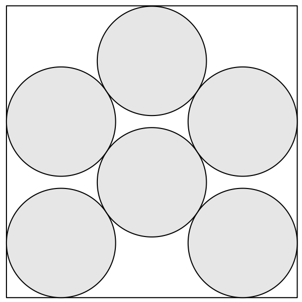
Christmas Invitational Mathematics Examination I
Friday, December 28, 2018

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET BEFORE STARTING YOUR TIMER.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.
3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators and computers are not permitted.**
4. A combination of the CIME and the Christmas Mathematics Contest 12 are not used to determine eligibility for participation in the Christmas Junior Mathematical Olympiad (CJMO). A combination of the CIME and the Christmas Mathematics Contest 10 are not used to determine eligibility for participation in the Christmas Junior Mathematical Olympiad (CJMO). All students are eligible to participate in the Christmas Junior Mathematical Olympiad. The CJMO will be given from FRIDAY, January 4, 2019 to FRIDAY, January 25, 2019.
5. You may obtain an AIME answer form from `https://www.maa.org/math-competitions/aime-archive`, and record all of your answers, and certain other information, on the AIME answer form. The answer form will not be collected from you, only your submission on the AIME Submission Form found at `https://artofproblemsolving.com/community/c594864h1747367`.

1. Find the largest positive integer less than 1000 with the property that both its base-10 and base-8 representation are palindromes.

2. Six circles of radius 1 are packed within a square of side length ℓ as shown. Adjacent circles are tangent to each other, while the five outer circles are tangent to the sides of the square. If $\ell = \frac{a}{\sqrt{b}} + c$ for positive integers a, b, c where b is not divisible by the square of any prime, compute $a + b + c$.



3. The increasing sequence

$$12,\ 15,\ 18,\ 21,\ 51,\ 81, \ldots$$

consists of all the positive multiples of 3 containing at least one digit that is a 1. The number 2019 is the N^{th} term of this sequence, where N is a positive integer. Find N.

4. Let $\{a_n\}$ be a sequence of positive integers such that $a_1 = a_2 = 1$ and $a_n = a_{n-1} + a_{n-2} + \gcd(a_{n-1}, a_{n-2})$ for all $n \geq 3$. The value of

$$\sum_{k=2}^{\infty} \frac{1}{a_{k+1} - a_k}$$

can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Find $m + n$.

5. Let k be a real number such that $\log_9(a), \log_9(b), \log_9(c)$ are the roots of $p(x) = x^3 - kx^2 + kx - 1$. If $a + b + c = 93$, then $k = \log_9(n)$ for a positive integer n. Find the remainder when n is divided by 1000.

6. Let $1 = d_1 < d_2 < \cdots < d_k = n$ be the divisors of a positive integer n. Suppose that $d_3^2 + d_4^2 = 2n + 1$. Find the sum of all possible values of n.

7. Albert, Bob, Carrie, and Douglas are travelling along a road at constant not necessarily equal velocities. Albert meets Bob at 12:00 pm, Carrie at 12:20 pm and Douglas at 12:32 pm. Later that same day, Douglas meets Carrie at 12:53 pm. and Bob at 1:17 pm. If Bob and Carrie meet m minutes after noon, compute m.

8. In parallelogram $ABCD$, the circumcircle of $\triangle BCD$ has center O and intersects lines AB and AD at E and F, respectively. Let P and Q be the midpoints of AO and BD, respectively. Suppose that $PQ = 3$ and the height from A to BD has length 7. Find the value of $BF \cdot DE$.

9. Let N denote the number of strictly increasing sequences of positive integers $a_1, a_2, \cdots, a_{19}$ satisfying the following two rules:

   - $a_1 = 1$ and $a_{19} = 361$,
   - for any $i \neq j$, if b_{ij} is the $(i \cdot j)^{\text{th}}$ number not in the sequence, then $(a_i - b_{ij})(a_j - b_{ij}) < 0$.

   Find the largest positive integer k such that 2^k divides N.

10. Let x, y, and z be real numbers such that $x^2 = 3 - 2yz, y^2 = 4 - 2xz$, and $z^2 = 5 - 2xy$. The value of $x^2 + y^2 + z^2$ can be written as $a + \frac{\sqrt{b}}{c}$ for positive integers a, b, c, where $\gcd(a, c) = 1$. Find $a + b + c$.

11. We define a positive integer to be *multiplicative* if it can be written as the sum of three distinct positive integers x, y, z such that y is a multiple of x and z is a multiple of y. Find the sum of all the positive integers which are not *multiplicative*.

12. Let $\mathcal{T}$ be the locus of all points z in the complex plane satisfying $|\mathrm{Re}(z)| = \mathrm{Im}(z) + 1$, and let $\mathcal{T}'$ be the locus of all points $z' = i/z$ where $z \in \mathcal{T}$ and $i = \sqrt{-1}$. If the area enclosed by $\mathcal{T}'$ is $\mathcal{A}$, compute $\lfloor 100\mathcal{A} \rfloor$.

13. Suppose P is a monic polynomial whose roots a, b and c are real numbers that satisfy the relation

$$a(a - b) = b(b - c) = c(c - a) = 1.$$

Find the greatest integer less than or equal to $100|\mathrm{P}(\sqrt{3})|$.

14. Let $\triangle ABC$ be a triangle with circumcenter O and incenter I such that the lengths of the three segments AB, BC and CA form an increasing arithmetic progression in this order. If $AO = 60$ and $AI = 58$, then the distance from A to BC can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

15. Let $\{\mathcal{F}_n\}_{n \geq 1}$ be a sequence of functions going from $\mathbb{N}^+$ to $\mathbb{N}^+$ defined recursively by $\mathcal{F}_1(n) = 1$ and

$$\mathcal{F}_k(n) = \sum_{d|n} \mathcal{F}_{k-1}(d)$$

for all $k > 1$. Compute the greatest integer less than or equal to $\frac{\mathcal{F}_{2019}(864)}{\mathcal{F}_{2019}(648)}$.