MAC CMC

Christmas Mathematics Competitions

CHRISTMAS MATHEMATICS COMPETITIONS

2ND ANNUAL

CIME II

SOLUTIONS PAMPHLET

CHRISTMAS INVITATIONAL MATHEMATICS EXAMINATION II SOLUTIONS PAMPHLET

FRIDAY, FEBRUARY 22, 2019

This Solutions Pamphlet gives at least one solution for each problem on this year's CIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

Correspondence about the problems and solutions for this CIME and orders for any of the publications listed below should be PM'd to:

AOPS12142015, eisirrational, FedeX333X, and TheUltimate123.

The problems and solutions for this CIME were prepared by the CMC's Committee on the CIME under the direction of:

Kyle Lee	Eric Shen
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1. (Answer: 187)

   We claim that no scalene triangle works. Assume for the sake of contradiction that a triangle with sides $F_a < F_b < F_c$ works. Then,

   $$F_a + F_b \leq F_{b-1} + F_b = F_{b+1} \leq F_c,$$

   a contradiction. Hence, the triangle is isosceles. If the leg has length $F_2 = 1$, then only F_2 works for the base. If $i > 2$ and the leg has length F_i, since

   $$2F_i > F_{i-1} + F_i = F_{i+1}, \text{ but } 2F_i < F_i + F_{i+1} = F_{i+2},$$

   the other leg must be $\{F_2, F_3, \dots, F_{i+1}\}$. However, for $i = 2019$, F_{2020} is not in $\mathcal{F}$, so the requested remainder is

   $$\begin{aligned} 1 + 3 + 4 + \cdots + 2017 + 2018 + 2018 &\equiv \frac{2019 \cdot 2020}{2} - 3 \\ &\equiv 19 \cdot 10 - 3 \\ &\equiv 187 \pmod{1000}, \end{aligned}$$

   and we are done.

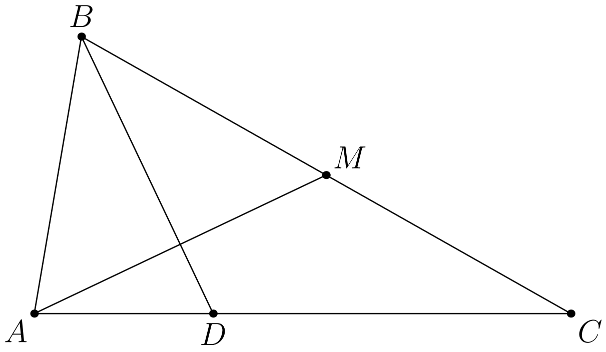
2. (Answer: 029)

   Since one of the products must be 0, there must be either 3 or 4 products that are negative, and thus 1 and 0 products that are positive as a result, respectively. We find the number of partitions that work by taking cases:

   - *Case 1:* Three products are negative. Then, one is 0 and another is positive. Consider the pair with positive product. If the elements of this pair are positive, then there are two positive numbers remaining. Since we need three products to be negative, and each negative product must contain a positive element, this is a contradiction. Hence, the pair with positive product contains negative elements. Then, we can choose this pair in $\binom{5}{2} = 10$ ways. We now have 3 negatives, 4 positives, and 1 zero remaining. The 4 positives must be matched with the 3 negatives and 1 zero, so this can be done in $4! = 24$ ways. Then, there are 240 partitions in this case.
   - *Case 2:* Four products are negative. Then, each positive number must be matched with a negative number. This can be done in $5 \cdot 4 \cdot 3 \cdot 2 = 120$ ways. Then, the 0 must be matched with the remaining number.

   Hence, the number of successful partitions is $120 + 240 = 360$. The number of ways to select the pairs is $9!! = 945$, and so the desired probability is $\dfrac{360}{945} = \dfrac{8}{21}$. The requested sum is $8 + 21 = 29$.

3. (Answer: 285)



Notice that $\overline{BD}$ is both the B-altitude and the B-angle bisector of $\triangle ABM$, so $BA = BM$ and $AB : BC = 1 : 2$. Furthermore, by the Angle Bisector Theorem, $AD : DC = 1 : 2$. It follows that

$$[ABC] = 3 \cdot [ABD] = \frac{3}{2} \cdot [ABMD] = \frac{3}{2} \cdot \frac{20 \cdot 19}{2} = 285,$$

the answer.

4. (Answer: 085)

Let the RHS be $f(n)$, so we want $n = f(n)$. Clearly, $1^1 = 1_4, 2^2 = 10_4, 3^3 = 123_4$. We do casework on the number of digits of n in base 4.

- *Case 1:* n has 1 digit. It is easy to see that only 1 works.
- *Case 2:* n has 2 digits. First, we cannot have the digit 3, as 123_4 has three digits. If both the digits are 1's, then $f(n) = f(11_4) = 2_4$, which does not work. If there is one 1 and one 2, then $f(n) = 11_4$, which does not work, and if both are 2's, then $f(n) = f(22_4) = 20$, which does not work. Hence, there are no 2-digit Munchausen numbers.
- *Case 3:* n has 3 digits. Since $f(222_4) = 100_4$, we must have at least one 3. However, if we have two 3's, then $f(n) \geq 313_4$, but out of $313_4, 323_4, 332_4, 333_4$, only 313_4 works. Then, assume we have exactly one 3. If we have at least one 2, then $200_4 \leq f(n) \leq 203_4$, a contradiction. Then, we must have one 3 and two 1's, so $f(n) = 131_4$, which satisfies the conditions.
- *Case 4:* n has 4 digits. For 4-digit numbers, there must be at least three 3's, but if the remaining digit is $1, 2, 3$, so $f(n)$ is $1102_4, 1111_4, 1230_4$; none work.
- *Case 5:* n has $k \geq 5$ digits. Then, $n > 4_{10}^{k-1} > 3_{10}^3 k \geq f(k)$, so there are no Munchausen numbers with more than 4 digits.

The only base-4 Munchausen numbers are $1_4, 131_4, 313_4$, giving $1 + 29 + 55 = 85$.

5. (Answer: 047)

Notice that

$$\begin{aligned}\left(z+\frac{1}{z}\right)+\left(\omega+\frac{1}{\omega}\right)&=\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\\&=\frac{a}{b}+\frac{b}{a}+ab+\frac{1}{ab}\\&=\left(ab+\frac{1}{ab}\right)+\left(\frac{a}{b}+\frac{b}{a}\right).\end{aligned}$$

Similarly,

$$\begin{aligned}\left(z+\frac{1}{z}\right)\left(\omega+\frac{1}{\omega}\right)&=\left(a^2+\frac{1}{a^2}\right)+\left(b^2+\frac{1}{b^2}\right)\\&=\frac{a^2+b^2}{a^2b^2}+a^2+b^2\\&=\left(1+\frac{1}{a^2b^2}\right)(a^2+b^2)\\&=\left(ab+\frac{1}{ab}\right)\left(\frac{a}{b}+\frac{b}{a}\right).\end{aligned}$$

By Vieta's, WLOG let $z+\frac{1}{z}=ab+\frac{1}{ab}$ and $\omega+\frac{1}{\omega}=\frac{a}{b}+\frac{b}{a}$. Then, $z\in\{ab,\frac{1}{ab}\}$ and $\omega\in\{\frac{a}{b},\frac{b}{a}\}$. Notice that

$$ab=(5+2i)(18+13i)=(90-26)+(65+36)i=64+101i$$

and

$$\frac{b}{a}=\frac{18+13i}{5+2i}=\frac{(18+13i)(5-2i)}{29}=\frac{116+29i}{29}=4+i.$$

It is easy to see that the maximum possible value of $|z+\omega|$ occurs when $z=ab$ and $\omega=\frac{b}{a}$. Hence,

$$\max(|z+\omega|)=|68+102i|=34\,|2+3i|=34\sqrt{13},$$

and the requested sum is $34+13=47$.

6. (Answer: 974)

Let $\mathbb{P}(x,y)$ be the probability the frog reaches (x,y). Then, it is easy to see that

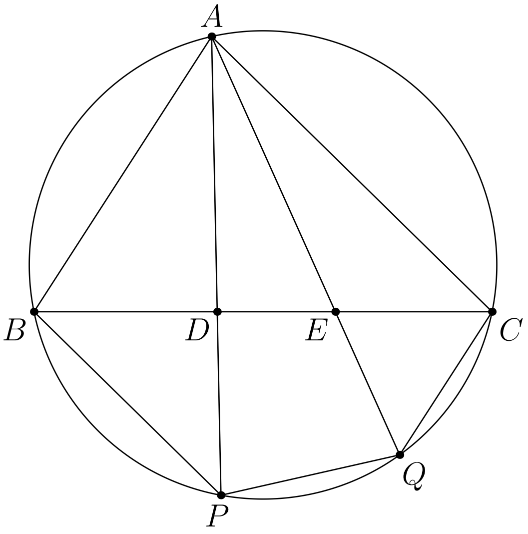
$$\mathbb{P}(x,y)=\frac{\mathbb{P}(x-1,y)+\mathbb{P}(x,y-1)+\mathbb{P}(x-1,y-1)}{3},$$

where $\mathbb{P}(x,0)=\mathbb{P}(0,x)=1/3^x$. We can use dynamic programming to evaluate $\mathbb{P}(3,3)$:

$\frac{1}{27}$	$\frac{13}{81}$	$\frac{73}{243}$	$\frac{245}{729}$
$\frac{1}{9}$	$\frac{1}{3}$	$\frac{11}{27}$	$\frac{73}{243}$
$\frac{1}{3}$	$\frac{5}{9}$	$\frac{1}{3}$	$\frac{13}{81}$
1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$

Thus, $\mathbb{P}(3,3) = \dfrac{245}{729}$, and the requested sum is $245 + 729 = 974$.

7. (Answer: 019)



Solution 1. Since $\angle ADB = \angle AOB = 2C$, $DA = DC$. Similarly, $EA = EB$, so $ABPC$ and $ABQC$ are isosceles trapezoids. Moreover, $AP = AQ = BC = 7$, $AB = CP = 5$, and $AC = BQ = 6$. Now, by Ptolemy's Theorem on $ABPC$,

$$AB \cdot CP + AC \cdot BP = BC \cdot AP \implies 5^2 + 6BP = 7^2 \implies BP = 4.$$

Similarly, by Ptolemy's Theorem on $ABQC$,

$$AB \cdot CQ + AC \cdot BQ = BC \cdot AQ \implies 5CQ + 6^2 = 7^2 \implies CQ = \frac{13}{5}.$$

Finally, by Ptolemy's on $BPQC$,

$$BC \cdot PQ + BP \cdot CQ = BQ \cdot CP \implies 7PQ + \frac{52}{5} = 5 \cdot 6 \implies PQ = \frac{14}{5},$$

and the requested sum is $14 + 5 = 19$.

Solution 2. Since $\angle ADB = \angle AOB = 2C$, $DA = DC$, and similarly, $EA = EB$. Furthermore, $AP = AQ = BC = 7$. Now, by Heron's on $\triangle ABC$, since the semiperimeter is 9,

$$[ABC] = \sqrt{9 \cdot 2 \cdot 3 \cdot 4} = 6\sqrt{6}.$$

It follows that the circumradius is

$$R = \frac{5 \cdot 6 \cdot 7}{4 \cdot 6\sqrt{6}} = \frac{35}{4\sqrt{6}}.$$

Notice that

$$\angle PAQ = \angle BAE + \angle DAC - \angle BAC = B + C - A = 180 - 2A,$$

whence $\angle APQ = A$. Then, by the Law of Cosines,

$$\cos A = \frac{5^2 + 6^2 - 7^2}{2 \cdot 5 \cdot 6} = \frac{1}{5},$$

so

$$PQ = 2AP\cos A = \frac{14}{5},$$

and the requested sum is $14 + 5 = 19$.

8. (Answer: 695)

Notice that for every anti-wavy integer with $a \neq 9$, $9-a < 9-b > 9-c < 9-d > 9-e$ produces a wavy integer, so we only need to find the number of quadruples of digits (b, c, d, e) such that $b < c > d < e$. (Note that since $b < c$, the condition that $9 > b$ is irrelevant.)

We first fix c. Then, since $0 \leq b, d < c$, they can been chosen is c^2 ways. After fixing d, since $d < e$, we can choose e in $9-d$ ways, so the number of ways to choose b, d and e is

$$c\sum_{d=0}^{c-1}(9-d) = c\left(9c - \frac{c(c-1)}{2}\right) = \frac{19c^2 - c^3}{2},$$

and the answer is

$$\begin{aligned}\sum_{c=1}^{9} \frac{19c^2 - c^3}{2} &= \frac{1}{2}\left(19 \cdot \frac{9 \cdot 10 \cdot 19}{6} - \frac{9^2 \cdot 10^2}{4}\right) \\ &= \frac{15}{2}(19^2 - 3 \cdot 45) \\ &= 1695,\end{aligned}$$

The requested remainder is 695.

9. (Answer: 206)

Lemma. The sum of the areas of all the rectangles in an $n \times n$ grid is $\binom{n+2}{3}^2$.

Proof. Consider each $1 \times n$ grid. Construct a unit square directly to the left and one to the right of the $1 \times n$ grid. Then, every $1 \times k$ rectangle of this subgrid can be bounded by two "boundary" squares that are not in the rectangle, but everything in between is in the rectangle. Then, the sum of the areas of the $1 \times k$ rectangles is the number of unit squares between two boundary squares. This can be done in $\binom{n+2}{3}$ ways. The desired result follows. □

Call a rectangle *dark* is it has more black squares than white and *bright* if it has more white squares than black. It is easy to check that a square is dark iff all its corners are black and a square is bright iff all its corners are white. Furthermore, a dark rectangle has one more black square than white, and vice versa. Now, note that translating a dark rectangle one unit right maps it to a light rectangle, so we only need to compute the average area of a rectangle in $\mathcal{S}$, which is $\binom{21}{3}^2$, the number of dark rectangles whose right side coincides with that of the 19×19 square, which is $10\binom{11}{2}$ by Stars and Bars, and the number of light rectangles whose left side coincides with that of the 19×19 square, which is $10\binom{10}{2}$ by Stars and Bars. The answer is

$$\left(\frac{1}{2}\binom{21}{3}^2 + \frac{10}{2}\binom{11}{2} - \frac{10}{2}\binom{10}{2}\right) \div \binom{20}{2}^2 = \frac{8845}{361},$$

and the answer is $8845 + 361 = 9206$.

10. (Answer: 508)

Solution 1. Let this product be $\mathcal{P}$. Then,

$$\mathcal{P} = \prod_{n=1}^{2019}(n^n + n) = 2019! \prod_{n=1}^{2019}(n^{n-1} + 1).$$

Now, we just need to find the number of factors of 3 in $n^{n-1} + 1$. However, if $3 \mid n^{n-1} + 1$, $n^{n-1} \equiv 2 \pmod 3$. Since $\operatorname{lcm}(3, \varphi(3)) = 6$, we just need to check all residues modulo 6. Indeed, $n^{n-1} \equiv 2 \pmod 3$ if and only if $n \equiv 2 \pmod 6$. Furthermore, if $n \equiv 2 \pmod 6$, notice that

$$n^{n-1} + 1 = (n+1)(n^{n-2} - n^{n-3} + \cdots + 1).$$

But,

$$n^{n-2} - n^{n-3} + \cdots + 1 \equiv 1 - (-1) + \cdots + 1 \equiv n - 1 \not\equiv 0 \pmod 3,$$

so the answer is

$$\begin{aligned}\nu_3(\mathcal{P}) &= \nu_3\left(2019!\prod_{n=1}^{2019}(n^{n-1}+1)\right)\\ &= \nu_3(2019!) + \nu_3\left(\prod_{\substack{i\le 2019,\text{ and}\\ i\equiv 2 \pmod 6}}(i+1)\right)\\ &= 1005 + \nu_3\left(\prod_{i=1}^{337}(6i-3)\right)\\ &= 1005 + \nu_3(3^{337}\cdot 673!!)\\ &= 1005 + 337 + \nu_3(673!) - \nu_3\left(\left\lfloor\frac{673}{2}\right\rfloor!\right)\\ &= 1005 + 337 + 332 - 166\\ &= 1508,\end{aligned}$$

and the requested remainder is 508.

Alternate Solution. As above, write

$$\mathcal{P} = \prod_{n=1}^{2019}(n^n+n) = 2019!\prod_{n=1}^{2019}(n^{n-1}+1).$$

Also, notice that $n^{n-1}+1$ is divisible by 3 if and only if $n \equiv 2 \pmod 6$. However, by LTE, if $n \equiv 2 \pmod 6$, $\nu_3(n^{n-1}+1) = \nu_3(n+1) + \nu_3(n-1) = \nu_3(n+1)$, and we can finish like above.

11. (Answer: 121)

To begin, let L' be the reflection of A over the perpendicular bisector of $\overline{BC}$, so that $\triangle ABC \cong \triangle L'CB$. However, by Ptolemy's on $ABCL'$,

$$5AL' + 4\cdot 4 = 6\cdot 6 \implies AL' = 4.$$

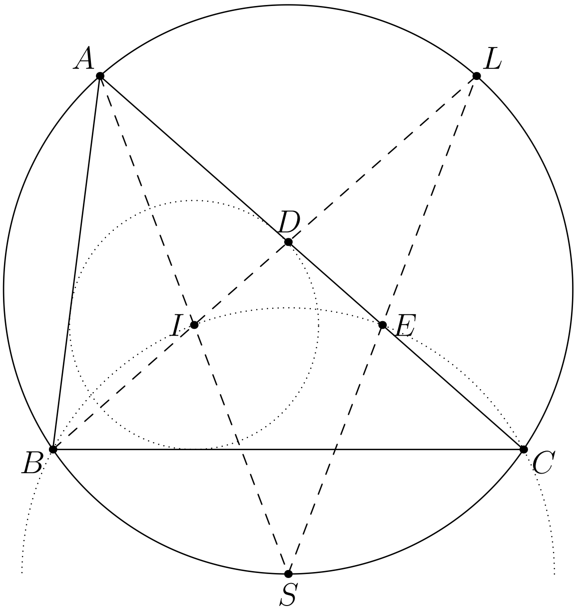
However, it follows that $AB = AL' = CL' = 4$, so $L = L'$; that is, $ABCL$ is an isosceles trapezoid. Now, by the Incenter-Excenter Lemma, $SB = SI = SC = d$ for some d. By Ptolemy's on $ABSC$,

$$4\cdot d + 6\cdot d = 10\cdot AS \implies d = \frac{AS}{2},$$

whence I is the midpoint of $\overline{AS}$.

The semiperimeter s of $\triangle ABC$ is clearly $15/2$. It follows by Heron's and $K = rs$, where r denotes the inradius of $\triangle ABC$, that

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \sqrt{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\div\frac{15}{2}} = \frac{\sqrt{7}}{2},$$



If F denotes the tangency point between the incircle and $\overline{AB}$, we can compute that $AF = s - a = 5/2$. Then, by the Pythagorean Theorem on $\triangle AFI$, $AI^2 = \sqrt{(s-a)^2 + r^2} = 2\sqrt{2}$.

Now, since I and E are the midpoints of $\overline{SA}$ and $\overline{SL}$, respectively, D is the centroid of $\triangle ASL$, so $[SIDE] = [ASL]/3$. However, since $AS = 2AI = 4\sqrt{2}$ and $AL = 4$, the distance from S to AL is $\sqrt{(4\sqrt{2})^2 - 2^2} = 2\sqrt{7}$, whence

$$[SIDE] = \frac{[ASL]}{3} = \frac{\frac{1}{2}\cdot 4\cdot 2\sqrt{7}}{3} = \frac{4\sqrt{7}}{3} \implies [SIDE]^2 = \frac{112}{9},$$

and the requested sum is $112 + 9 = 121$.

Alternate Solution. By the Angle Bisector Theorem, $AD = 8/3$ and $DC = 10/3$. However,

$$BD^2 = BA\cdot BC - DA\cdot DC = 20\left(1 - \left(\frac{2}{3}\right)^2\right) = \frac{100}{9},$$

so $\triangle BDC$ is isosceles. Then, $AL = LC = AB = 4$, and we may proceed as above.

12. (Answer: 349)

Suppose that Max attempts only one free throw a day. Then, the expected number of days it takes Max to complete level n is

$$\left(1 - \frac{1}{(n+1)^2}\right)^{-1} = \left(\frac{n(n+2)}{(n+1)^2}\right)^{-1} = 1 + \frac{1}{n(n+2)} = 1 + \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right).$$

Thus, the expected number of days it takes Max to complete all 99 levels is

$$\begin{aligned}\sum_{n=1}^{99}\left(1-\frac{1}{(n+1)^2}\right)^{-1} &= \sum_{n=1}^{99}\left(\frac{n(n+2)}{(n+1)^2}\right)^{-1}\\ &= 99+\frac{1}{2}\sum_{n=1}^{99}\left(\frac{1}{n}-\frac{1}{n+2}\right)\\ &= 99+\frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}-\frac{1}{100}-\frac{1}{101}\right)\\ &= 99+\frac{14949}{20200}.\end{aligned}$$

However, we overcount by 98 days, so the expected value of K is $1+\frac{14949}{20200}=\frac{35149}{20200}$, and the requested remainder is $35149+20200\equiv 349 \pmod{1000}$.

13. (Answer: 072)

Let $\mathcal{S}(n)$ denote the number of entries of an $n\times n$ times table that are divisible by n. We claim that

$$\mathcal{S}(n)=\sum_{d\mid n}\frac{\varphi(d)}{d}.$$

First, we take a positive integer $m\le n$. If $d=\gcd(m,n)$, then $m=dm'$ and $n=dn'$, where $\gcd(m',n')=1$. It follows that if $n\mid mu$, then $n'\mid m'u$, so $n'\mid u$. There are exactly d such u that are divisible by n', whence

$$\mathcal{S}(n)=\sum_{m=1}^{n}\gcd(m,n).$$

Now, we look at each divisor d of n. Like above, notice that $d=\gcd(m,n)$ if and only if $m=dm'$ and $n=dn'$, where $\gcd(m',n')=1$. However, the number of m with $\gcd(m,n)=1$ is $\varphi(n')=\varphi(d/n)$, so

$$\mathcal{S}(n)=\sum_{m=1}^{n}\gcd(m,n)=\sum_{d\mid n}d\cdot\varphi\left(\frac{n}{d}\right)=n\sum_{d\mid n}\frac{\varphi(d)}{d},$$

as required. A corollary of this claim is that $\mathcal{S}=\mathrm{Id}*\varphi$, where $*$ denotes Dirichlet convolution and $\mathrm{Id}(n)=n$. Since φ and Id are multiplicative, so is $\mathcal{S}$. Then, if $n=p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$,

$$\begin{aligned}\mathcal{S}(n)&=\prod_{i=1}^{k}\left(p_i^{e_i}\sum_{j=0}^{e_i}\frac{\varphi(p_i^j)}{p_i^j}\right)\\ &=\prod_{i=1}^{k}p_i^{e_i}\left(1+\frac{e_i(p_i-1)}{p_i}\right)\\ &=\prod_{i=1}^{k}p_i^{e_i-1}((p_i-1)(e_i+1)+1).\end{aligned}$$

Since $720 = 2^4 \cdot 3^2 \cdot 5$, $N = 48 \cdot 21 \cdot 9 = 9072$, and the requested remainder is 72.

14. (Answer: 049)

First let $x = \tan\alpha$, $y = \tan\beta$, and $z = \tan\gamma$. We claim that $xy + yz + zx = 1$. Notice that

$$z = \tan\gamma = \tan(90 - \alpha - \beta) = \frac{1 - xy}{x + y},$$

and the desired result readily follows.

Now, since $\tan^2\theta + 1 = \sec^2\theta$, $\sec\alpha = \sqrt{x^2+1}$, and so on. It follows that

$$xyz + \sqrt{(x^2+1)(y^2+1)(z^2+1)} = \frac{37}{20}.$$

However, if $i = \sqrt{-1}$ and $P(t)$ is the monic cubic with roots x, y, z,

$$\begin{aligned}(x^2+1)(y^2+1)(z^2+1) &= P(i)P(-i)\\ &= ((x+y+z) - xyz)^2.\end{aligned}$$

We claim that $x + y + z \geq xyz$. Assume FTSoC that $x + y + z < xyz$. Then,

$$\begin{aligned}x(yz-1) > y + z = (1 - yz)\tan(\beta+\gamma) &= \frac{1}{x}(1 - yz)\\ \implies \left(x + \frac{1}{x}\right)(yz - 1) &> 0.\end{aligned}$$

Since $x > 0$, $yz > 1$. Similarly, $zx > 1$ and $xy > 1$. However, this implies that

$$1 = xy + yz + zx > 3,$$

a contradiction. Now, we have that $x + y + z = \frac{37}{20}$, so

$$x^2 + y^2 + z^2 = (x+y+z)^2 - 2(xy + yz + zx) = \frac{569}{400}.$$

However, by the Cauchy-Schwarz Inequality,

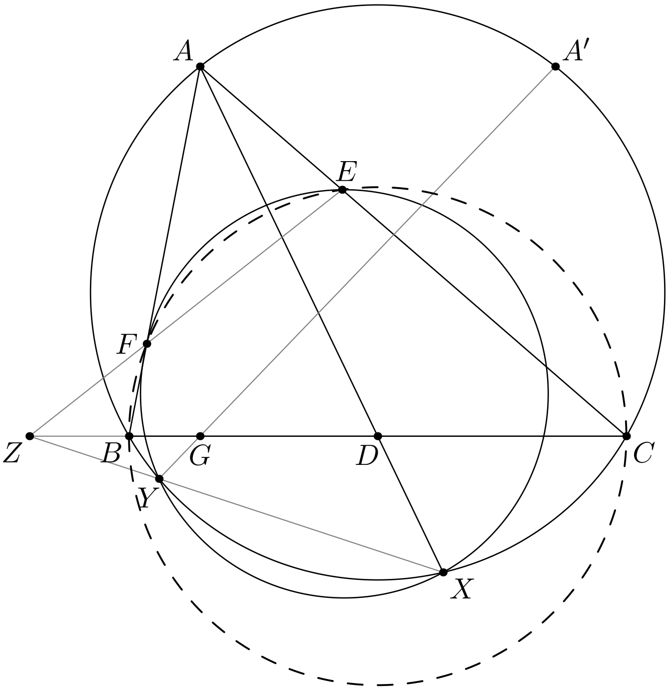
$$2\left(\frac{569}{400} - x^2\right) = 2(y^2 + z^2) \geq (y+z)^2 = \left(\frac{37}{20} - x\right)^2.$$

This solves to

$$\frac{11}{60} \leq x \leq \frac{21}{20}.$$

Since $\cos\alpha$ is minimized when $\tan\alpha$ is maximized, take $x = \frac{21}{20}$ to get $\cos\alpha = \frac{20}{29}$, and the requested sum is $20 + 29 = 49$.

15. (Answer: 265)



Let Ω be the circle with diameter $\overline{BC}$, so that $E, F \in \Omega$. Let $Z = \overline{BC} \cap \overline{EF}$ and A' be the second intersection of $\overline{YG}$ with ω. First, notice that Z is the radical center of ω, Ω, and (EFX), so Z lies on $\overline{XY}$. Furthermore, since G, D, E, F lie on the nine-point circle of $\triangle ABC$,

$$ZX \cdot ZY = ZE \cdot ZF = ZG \cdot ZD,$$

and $GDXY$ is cyclic. Then,

$$\measuredangle AA'G = \measuredangle AA'Y = \measuredangle AXY = \measuredangle DXY = \measuredangle DGY = \measuredangle DGA',$$

whence $\overline{AA'} \parallel \overline{BC}$, so $\triangle ABC \cong \triangle A'CB$.

By the Law of Cosines on $\triangle ABC$,

$$BC^2 = AB^2 + AC^2 - AB \cdot AC = 28,$$

so $BC = 2\sqrt{7}$. Now, by the Law of Sines and the Ratio Lemma,

$$\frac{YC}{YB} = \frac{\sin \angle CA'G}{\sin \angle BA'G} = \frac{3}{2} \cdot \frac{GC}{GB} = 9.$$

Then, if $x = YB$, by the Law of Cosines on $\triangle YBC$,

$$28 = x^2 + (9x)^2 + x \cdot (9x) = 91x^2 \implies YB = \frac{2}{\sqrt{13}}.$$

Finally, by Ptolemy's Theorem on $ABYC$,

$$AY = \frac{1}{2\sqrt{7}} \left(\frac{4 \cdot 18}{\sqrt{13}} + \frac{6 \cdot 2}{\sqrt{13}} \right) = \frac{6\sqrt{7}}{\sqrt{13}} \implies AY^2 = \frac{252}{13},$$

and the requested sum is $252 + 13 = 265$.

Alternate Solution. Like above, remark that $\overline{BC}, \overline{EF}, \overline{XY}$ concur at Z. Notice that

$$(C, B; Y, A) \overset{X}{=} (C, B; Z, D).$$

Since $-1 = (B, C; G, Z)$ by Ceva-Menelaus,

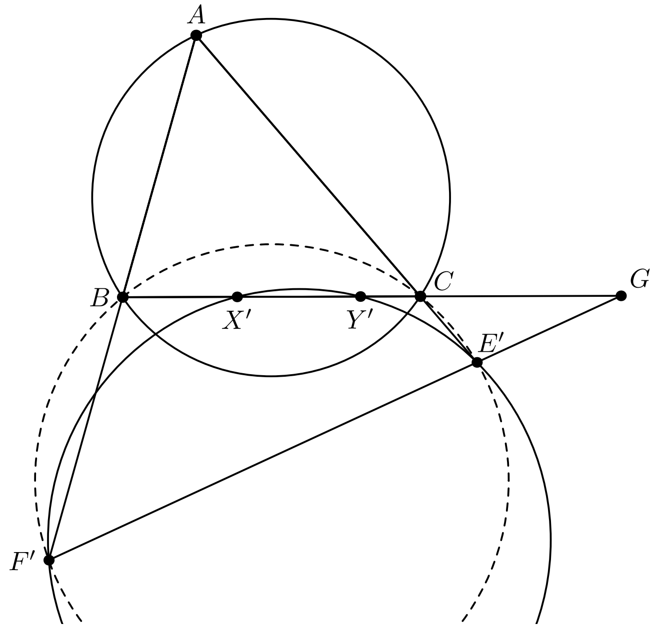
$$\frac{YC}{YB} = \frac{AC}{AB} \cdot \frac{ZC}{ZB} \cdot \frac{DB}{DC} = \frac{3}{2} \cdot \frac{GC}{GB} = 9,$$

and we may finish like above.

Solution 2. Scale down by a factor of 2 so that $AB = 2$ and $AC = 3$. Perform an inversion about the circle with center A and radius $\sqrt{AB \cdot AC}$ followed by a reflection about the angle bisector of $\angle A$. Then X is taken to the foot X' of the A-symmedian. Furthermore, E is taken to a point E' such that

$$AE' = \frac{AB \cdot AC}{AE} = \frac{AC}{\cos 60^\circ} = 2AC = 6,$$

and similarly $AF' = 4$; this implies $BE' = 4$ and $CF' = 1$. Then Y' is taken to the intersection of $\odot(X'E'F')$ and BC.



Let $G = E'F' \cap BC$, and set $a := BC = \sqrt{7}$ for convenience. Compute via Menelaus on $\triangle AE'F'$ and transversal $\overline{BCG}$ that $CG = \frac{3}{8}a$. Now remark that by Steiner $\frac{BX'}{X'C} = (\frac{2}{3})^2 = \frac{4}{9}$, so $X'C = \frac{9}{13}a \Rightarrow GX' = \frac{84}{65}a$. Further noting that BC and $E'F'$ are antiparallel (so that $BCE'F'$ is in fact cyclic), we may apply Power of a Point to obtain $GB \cdot GC = GE' \cdot GF' = GX' \cdot GY'$, which implies

$$GY' = \frac{GB \cdot GC}{GX'} = \frac{(\frac{3}{5}a)(\frac{8}{5}a)}{\frac{84}{64}a} = \frac{26}{35}a;$$

in turn, $CY' = GY' - GC = \frac{1}{7}a$.

Finally, a quick application of Stewart yields $AY' = \frac{2\sqrt{91}}{7}$, so

$$AY = \frac{AB \cdot AC}{AY'} = \frac{6}{\frac{2\sqrt{91}}{7}} = \frac{3\sqrt{91}}{13}.$$

Scaling up yields $AY^2 = (\frac{6\sqrt{91}}{13})^2 = \frac{252}{13}$, so the requested sum is $252 + 13 = 265$.