The 2nd CIME II Solutions Pamphlet will be released after the contest.

CONTACT US – *Correspondence about the problems and solutions for this CIME should be sent by PM to:*

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The problems and solutions for this AIME were prepared by the CMC's Committee on the CIME under the direction of:

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2019 CJMO – THE CHRISTMAS MATHEMATICAL JUNIOR OLYMPIAD (CJMO) is a 6-question, 9-hour, essay-type examination. The CJMO will be held from Friday, January 4, 2019 to Friday, January 25, 2018. Your teacher will not have more details on who qualifies for the CJMO in the CMC 10/12 and CIME Teachers' Manuals because we did not make Teachers' Manuals and all students are qualified for the CJMO. The best way to prepare for the CJMO is to study previous years of these exams, which may be found on our website as indicated below.

PUBLICATIONS – For a complete listing of our previous competitions please visit our website at `https://sites.google.com/view/annualcmc/`.

The

MAC Christmas Mathematics Competitions

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MAC CMC

Christmas Mathematics Competitions

CHRISTMAS MATHEMATICS COMPETITIONS

2ND ANNUAL

CIME II

CHRISTMAS INVITATIONAL MATHEMATICS EXAMINATION II

FRIDAY, FEBRUARY 22, 2019

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET BEFORE STARTING YOUR TIMER.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.
3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators and computers are not permitted.**
4. A combination of the CIME and the Christmas Mathematics Contest 12 are not used to determine eligibility for participation in the Christmas Junior Mathematical Olympiad (CJMO). A combination of the CIME and the Christmas Mathematics Contest 10 are not used to determine eligibility for participation in the Christmas Junior Mathematical Olympiad (CJMO). All students are eligible to participate in the Christmas Junior Mathematical Olympiad. The CJMO will be given from FRIDAY, January 4, 2019 to FRIDAY, January 25, 2019.
5. You may obtain an AIME answer form from `https://www.maa.org/math-competitions/aime-archive`, and record all of your answers, and certain other information, on the AIME answer form. The answer form will not be collected from you, only your submission on the AIME Submission Form found at `https://artofproblemsolving.com/community/c594864h1747367`.

1. Consider the sequence $\{F_n\}$ defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n > 2$. Suppose that $\mathcal{F} = \{F_2, F_3, \ldots, F_{2019}\}$. Then, there are T non-congruent triangles with positive area whose side lengths are all (not necessarily distinct) elements of $\mathcal{F}$. Find the remainder when T is divided by 1000.

2. Consider the set $S = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4\}$. We partition this set into 5 pairs, and the product of the two numbers in each pair is computed. The probability that at least half of the products are negative can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

3. In triangle ABC, the median $\overline{AM}$ is perpendicular to the angle bisector $\overline{BD}$. Suppose that $AM = 20$ and $BD = 19$. Find the area of $\triangle ABC$.

4. Define a base-b Munchausen number as a positive integer n such that when expressed in base b, its digits $d_0 \cdots d_k$ are all nonzero and satisfy
$$n = \sum_{i=0}^{k} d_i b^i = \sum_{i=0}^{k} d_i^{d_i}.$$
For instance, 3435 is a base-10 Munchausen number, since $3^3 + 4^4 + 3^3 + 5^5 = 3435$. Find, in base 10, the sum of all base-4 Munchausen numbers.

5. Let $a = 5 + 2i$ and $b = 18 + 13i$, where $i = \sqrt{-1}$. Suppose that z and ω are complex numbers such that
$$\left(z + \frac{1}{z}\right) + \left(\omega + \frac{1}{\omega}\right) = \left(a + \frac{1}{a}\right) \times \left(b + \frac{1}{b}\right), \text{ and}$$
$$\left(z + \frac{1}{z}\right) \times \left(\omega + \frac{1}{\omega}\right) = \left(a^2 + \frac{1}{a^2}\right) + \left(b^2 + \frac{1}{b^2}\right).$$
Then, the largest possible value of $|z + \omega|$ can be expressed as $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

6. A frog is initially at the point $(0, 0)$. Every second, the frog jumps from its current position (x, y) to either $(x+1, y)$, $(x, y+1)$, or $(x+1, y+1)$, each with equal probability. The probability that the frog eventually reaches the point $(3, 3)$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

7. In $\triangle ABC$, $AB = 5$, $AC = 6$, and $BC = 7$. Let point O be the circumcenter of $\triangle ABC$ and let the circumcircles of $\triangle AOB$ and $\triangle AOC$ intersect $\overline{BC}$ again at D and E, respectively. Rays AD and AE meet the circumcircle of $\triangle ABC$ again at P and Q, respectively. Then, $PQ = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

8. A 5-digit positive integer $\overline{abcde}$ is defined to be *wavy* if $a < b > c < d > e$ and *anti-wavy* if $a > b < c > d < e$. Find the remainder when the positive difference between the number of wavy integers and the number of anti-wavy integers is divided by 1000.

9. Let $\mathcal{S}$ be the set of all rectangles formed by the squares of a 19×19 checkerboard whose corners are black. A rectangle's darkness is the number of black squares within its interior. The average darkness of a rectangle in $\mathcal{S}$ can be written as $\frac{m}{n}$, where m and n are relatively prime integers. Find the remainder when $m + n$ is divided by 1000.

10. Let N be the exponent of 3 in the prime factorization of
$$(1^1 + 1)(2^2 + 2)(3^3 + 3) \cdots (2019^{2019} + 2019).$$
Find the remainder when N is divided by 1000.

11. In triangle ABC with incenter I, $AB = 4$, $BC = 5$, and $CA = 6$. If lines AI and BI meet the circumcircle of $\triangle ABC$ again at S and L, respectively, and $\overline{LB}$ and $\overline{LS}$ intersect $\overline{AC}$ at D and E, respectively, then the square of the area of quadrilateral $SIDE$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

12. Max is playing a video game with 99 levels, labeled $1, 2, \ldots, 99$. Whenever Max completes a level, he begins the next one immediately. However, for all $1 \le n \le 99$, Max fails the n^{th} level with probability $(n+1)^{-2}$. Whenever he fails a level, he quits for the day and attempts the level again the next day. If Max first attempts the first level on Day 1 and completes the 99^{th} level on Day K, then the expected value of K can be expressed as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find the remainder when $p + q$ is divided by 1000.

13. The $n \times n$ multiplication table is the $n \times n$ table of values where the entry in the i^{th} row and j^{th} column corresponds to the value ij. Let N be the number of entries in the 720×720 multiplication table that are divisible by 720. Find the remainder when N is divided by 1000.

14. Let $\alpha, \beta, \gamma > 0$ be angles such that $\alpha + \beta + \gamma = 90^\circ$ and
$$\tan\alpha \tan\beta \tan\gamma + \sec\alpha \sec\beta \sec\gamma = \frac{37}{20}.$$
Then, the minimum possible value of $\cos\alpha$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

15. In triangle ABC, $AB = 4$, $AC = 6$ and $\angle A = 60^\circ$. Define ω as the circumcircle of $\triangle ABC$, D as the midpoint of $\overline{BC}$, E as the foot of B onto $\overline{CA}$ and F as the foot of C onto $\overline{AB}$. Suppose $\overline{AD}$ intersects ω again at X and the circumcircle of $\triangle XEF$ intersects ω again at Y. Then, AY^2 can be expressed as $\frac{m}{n}$, where m and n are positive relatively prime integers. Find $m + n$.