

2nd Christmas Junior Mathematical Olympiad
Recommended Marking Scheme
January 4, 2019 – January 25, 2019

1. Call a convex equilateral polygon *rhomboidal* if it can be tiled with a finite number of non-overlapping rhombi that have the same side length of the polygon. Prove that a convex equilateral polygon is rhomboidal if and only if each side of the polygon is parallel to some other side of the polygon.

Solution. We will prove that an equilateral polygon \mathcal{P} is *rhomboidal* iff for each side ℓ_i there exists another side ℓ'_i with $\ell_i \parallel \ell'_i$; this also means that \mathcal{P} has an even number of sides.

- **Part 1:** If \mathcal{P} is *rhomboidal*, for each side ℓ_i there exists another side ℓ'_i with $\ell_i \parallel \ell'_i$.
Suppose that \mathcal{P} is *rhomboidal*: each side of \mathcal{P} must be a side of exactly one rhombus (if they were more than one, since the rhombi share the same side length of \mathcal{P} , the two or more rhombi would overlap). Let's start by placing a rhombus on the side ℓ_1 of \mathcal{P} , and consider the side $a_1 \parallel \ell_1$ of the rhombus: we place another rhombus on a_1 , and we continue placing rhombi on the side a_n of the n^{th} rhombus, where $a_1 \parallel a_2 \parallel \dots \parallel a_n$. Since we want to tile \mathcal{P} with a finite number of rhombi, this construction should end at some point, so one of the sides $a_i \parallel \ell$ of the rhombi must be a side ℓ'_1 of \mathcal{P} . Since this construction is valid for each side of \mathcal{P} , each side ℓ_i of \mathcal{P} is parallel to one (and one only since \mathcal{P} is convex) side ℓ'_i of the polygon. Hence our claim is proved; in particular, it follows that \mathcal{P} has an even number of sides.

2 points for proving the only if case. No partial credit should be awarded should this case be incomplete.

- **Part 2:** For each side ℓ_i if there exists another side ℓ'_i with $\ell_i \parallel \ell'_i$, then \mathcal{P} is *rhomboidal*.
We will prove the claim by induction. Clearly, a quadrilateral with opposite and parallel sides is a rhombus. Suppose now that all equilateral polygons \mathcal{P} of $2n$ sides with each side parallel to another side of the polygon are *rhomboidal*. Consider $n + 2$ consecutive vertices of an equilateral polygon \mathcal{P}' with $2n + 2$ sides: the first 3 such vertices identify two sides of a rhombus (see the diagram, in blue¹); drawing the other two sides and using the same construction of Part 1, we place other $n - 2$ rhombi (in light blue), each of them with two sides parallel to the one of the first rhombus, and this tiling closes on \mathcal{P}' because of our hypothesis of parallelism. Hence, we are left with an equilateral polygon with $2n$ sides, which by our inductive hypothesis, is *rhomboidal*. Hence, \mathcal{P}' is *rhomboidal*, and our claim has been proven. Done. \square

5 points for clear induction steps. The base case of a rhombus is negligible at the grader's discretion.

Remark. No points should be rewarded for simply mentioning induction and no points should be rewarded for mentioning the “only if” case is trivial. Possible scores for this problem are 0, 2, 5, 6, 7.

¹only in the official Solutions pdf

2. Prove that if a, b, c are real numbers, and the polynomial $P(x) = x^3 + ax^2 + bx + c$ has only real roots, then

$$(b - 1)^2 \leq \left(\frac{a^2 - 2b}{3} + 1 \right)^3,$$

and determine when equality occurs.

Solution. Let P 's roots be r_1, r_2, r_3 . Then, by AM-GM,

$$\left(\frac{a^2 - 2b}{3} + 1 \right)^3 = \left(\frac{(r_1^2 + 1) + (r_2^2 + 1) + (r_3^2 + 1)}{3} \right)^3 \geq (r_1^2 + 1)(r_2^2 + 1)(r_3^2 + 1).$$

1 point for showing that the RHS is equivalent to the above expression and noting that it suffices to prove the above is at least $(b - 1)^2$.

However,

$$(r_1^2 + 1)(r_2^2 + 1)(r_3^2 + 1) = P(i)P(-i) = (c - a)^2 + (b - 1)^2 \geq (b - 1)^2,$$

4 points are possible here. 3 points for showing that it is equal to $P(i)P(-i)$ and 1 point for showing that it is at least $(b - 1)^2$.

where equality holds iff $a = c$ and $|r_1| = |r_2| = |r_3|$. It is not hard to check that equality holds iff $(a, b, c) = (0, 0, 0), (3\sqrt{3}, 9, 3\sqrt{3}), (-3\sqrt{3}, 9, -3\sqrt{3})$.

2 points for arriving at these equality cases. Proof is not required. Deduct 1 point if one case is missing.

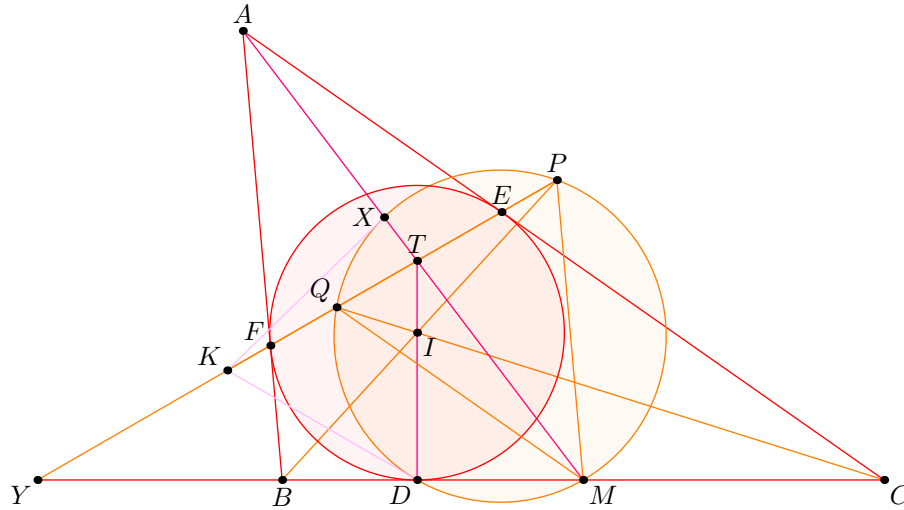
Remark. Explanation is not needed on how the contestant arrived at these expressions. Possible marks for this solution are 0, 1, 2, 4, 5, 6, 7.

3. (*) Let I be the incenter of $\triangle ABC$, and M be the midpoint of \overline{BC} . Let Ω be the nine-point circle of $\triangle BIC$. Suppose that \overline{BC} intersects Ω at a point $D \neq M$. If Y is the intersection of \overline{BC} and the A -intouch chord, and X is the projection of Y onto \overline{AM} , prove that X lies on Ω , and the intersection of the tangents to Ω at D and X lies on the A -intouch chord of $\triangle ABC$.

Note. The nine-point circle of $\triangle ABC$ is the circumcircle of its medial triangle, and if the incircle touches \overline{AC} and \overline{AB} at E and F , respectively, then \overline{EF} is the A -intouch chord.

First solution. Let the incircle of $\triangle ABC$ touch \overline{AC} and \overline{AB} at E and F , respectively. Furthermore, let $P = \overline{BI} \cap \overline{EF}$ and $Q = \overline{CI} \cap \overline{EF}$. By the Iran Lemma, $\angle BPC = \angle BQC = 90^\circ$, so $MP = MQ$. Let $T = \overline{AM} \cap \overline{EF}$. Obviously the incircle of $\triangle ABC$ touches \overline{BC} at D .

1 point for constructing the points P and Q such that $\angle BPC = \angle BQC$, $P, Q \in \overline{EF}$, $P \in \overline{BI}$, and $Q \in \overline{CI}$.



It is well-known that T lies on \overline{ID} . Then, by Ceva-Menelaus,

$$-1 = (B, C; D, Y) \stackrel{I}{=} (P, Q; T, Y).$$

However, by construction, $\angle TXY = 90^\circ$, so by a well-known lemma, \overline{XT} bisects $\angle PXQ$.

2 point for noticing that \overline{XT} bisects $\angle PXQ$.

Since $\triangle DPQ$ is the orthic triangle of $\triangle BIC$, $(DPQ) = \Omega$. However, because $MP = MQ$, M is the midpoint of \widehat{PQ} in Ω . By Apollonian circles, X is unique point on \overline{AM} such that \overline{XM} bisects $\angle PXQ$, whence $X \in (PMQ)$.

1 point for completing the first part.

Then, we have two ways to finish:

- *First approach.* Notice that

$$-1 = (P, Q; T, Y) \stackrel{M}{=} (P, Q; X, D),$$

and it follows that the intersection of the tangents to Ω at D and X lies on \overline{PQ} , which is the A -intouch chord, as required. \square

- *Second approach.* Since $(P, Q; T, Y)$ is harmonic, by the Midpoint of Harmonic Bundles Lemma, if K is the midpoint of \overline{YT} such that K is the circumcenter of $YDTX$,

$$KP \cdot KQ = KT^2 = KX^2 = KD^2,$$

and we are done. \square

3 points for completing the second part. For the first approach, showing that $(P, Q; X, D)$ is harmonic suffices. For the second approach, award no points for claiming that K is the desired intersection without proof.

Remark. This solution involves myriad configurations and therefore citing them as “well-known” suffices. The possible marks for this solution are 0, 1, 3, 4, 5, 7, with 1 point deductable for lack of a good diagram.

Second solution. Assume WLOG $\angle B > \angle C$. Clearly D is the point where the incircle touches \overline{BC} . Let \overline{EF} be the A -intouch chord, H be the orthocenter of $\triangle BIC$, and N and S be the midpoints of \overline{HI} and \overline{HC} , respectively. It is well-known that $\overline{AM}, \overline{EF}, \overline{ID}$ concur at a point, say T . Since $TX YD$ is cyclic,

$$\angle MXD = \angle TXD = \angle TYD = 180 - \angle CEY - \angle YCE = 90 - \frac{A}{2} - C.$$

However, if H_I and I_A denote the reflections of H over D and M , respectively, so that they lie on the circumcircle of $\triangle BIC$. If L is the intersection of the angle bisector of $\angle BIC$ with (BIC) , since $\widehat{H_I L} = \widehat{L I_A}$,

$$\angle MND = \angle I_A I H_I = 2\angle L I H_I = 2\angle C I D - 2\angle L I C = 90 - \frac{A}{2} - C,$$

and $X \in \Omega$.

4 points for the first part. Award only 1 point if only one of $\angle MXD$ and $\angle MND$ has been evaluated.

If K is the midpoint of \overline{YT} so that K is the circumcenter of $YDTX$, then

$$\angle KXD = 90 - \angle DYX = 90 - \angle DTX = 90 - \angle DTM = \angle TMD = \angle XMD,$$

so \overline{KX} is tangent to Ω . Furthermore, $KD^2 = KX^2$, so we are done. \square

3 points for finishing. Award no points for claiming that K is the desired intersection without proof.

Remark. Miraculously, this is a pure-angle chase solution to a problem that screams projective. The possible marks for this solution are 0, 1, 4, 7, with one point deductable for lack of a good diagram.

4. (*) Let ABC be a triangle with orthocenter H , and define E and F as the intersections of \overline{AH} with the perpendicular bisectors of \overline{AB} and \overline{AC} respectively. Furthermore, let D be the intersection of \overline{BE} and \overline{CF} . Suppose that X and Y lie on \overline{AB} and \overline{AC} respectively such that \overline{FX} , \overline{EY} , and \overline{BC} are all parallel. Prove that X and Y lie on the exterior angle bisector of $\angle BDC$.

Solution. Let O be the circumcenter of ABC , and define P and Q as the second intersections of \overline{BD} and \overline{CD} with the circumcircle of ABC . It follows that

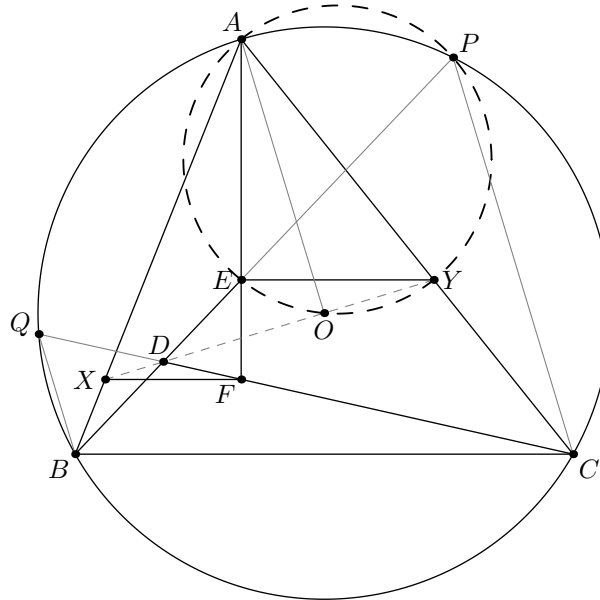
$$\widehat{QP} = 2(\angle FCA + \angle EBA) = 2(\angle FAC + \angle EAB) = 2\angle A = \widehat{BC},$$

$BQPC$ must be an isosceles trapezoid with bases parallel to \overline{AO} .

1 point for showing that $BQPC$ is an isosceles trapezoid.

From here, it follows that \overline{DO} is perpendicular to \overline{AO} and \overline{DO} bisects $\angle EDF$.

1 point for realizing that \overline{DO} is the external angle bisector of $\angle BDC$.



Deduct 1 point if no large, in-scale diagram is present.

Now, let Y' be the intersection of \overline{DO} with \overline{AC} . Trivial angle chasing yields that $\angle OY'A = \angle ABC = \angle OEA$, so $A, E, O,$ and Y' are concyclic and $\angle AEY' = \angle AOY' = 90^\circ$; hence, $Y = Y'$, and $AEOY$ is cyclic. Similarly, X lies on \overline{DO} , and $AFOX$ is cyclic. It follows that \overline{XY} is the external angle bisector of $\angle BDC$, as desired. \square

5 points for proving that X and Y lie on \overline{DO} .

Remark. This problem is pretty direct and is a sequence of steps, and therefore points are accumulative. Possible marks for this problem are 0, 1, 2, 7, with 1 point deductible for lack of a good diagram.

5. Let S be a set of $mn + 1$ points equally spaced around a circle. Exactly one line segment is drawn between every pair of points in S , and each line segment is colored one of m colors. Call a coloring of line segments *fair* if for any color C of the m colors and any point P in S , P is the endpoint of exactly n line segments of color C . Show that there exists a fair coloring for all positive integers m and n .

Solution. The answer is all (m, n) such that at least one of the following assertions is true:

- m is odd;
- n is even.

Award no points for stating this claim without proof.

First, we show that this is impossible if m is even and n is odd. Note that $mn + 1$ must then be odd. However, it follows that the edges of each color form a n -regular graph. Since the graph has an odd number of vertices, this is impossible by the Handshaking Lemma.

2 points for showing that m is even and n is odd leads to a contradiction. The Handshaking Lemma does not need to be explicitly mentioned.

A fair coloring clearly exists if n is even. Suppose the distance $d(P, Q)$ between two points P and Q on a circle is one more the number of points between them on the shortest path around the circle. In other words, the distance between two points is the smallest number of arcs one must pass through to get between the two points going around the circle. Label the colors C_1 to C_m . Then, for each pair of points P and Q , color the segment connecting P and Q color C_i if and only if

$$\frac{n}{2}(i - 1) < d(P, Q) \leq \frac{n}{2} \cdot i.$$

It is easy to see that this coloring is fair.

2 points for providing an example of a fair coloring if n is even.

Now, it suffices to show that if m and n are both odd, a fair coloring exists. We have two ways to finish.

- *First approach.* From here on, assume that m and n are odd. Label the $mn + 1$ points

$$V_0, V_{1,1}, V_{1,2}, \dots, V_{1,m}, V_{2,1}, \dots, V_{n,m}.$$

Furthermore, for all $1 \leq i \leq n$, let G_i be the complete graph containing the points $V_{i,1}, V_{i,2}, \dots, V_{i,m}$. We color G_i in a manner such that no two adjacent edges are the same color. Suppose that the vertices of G_i are arranged in a regular n -gon in the order $V_{i,1}, V_{i,2}, \dots, V_{i,m}$. Now, for every side of the polygon, if the opposite vertex is $V_{i,j}$, color the side C_j . Furthermore, color every diagonal the same color as the side parallel to it.

1 point for the coloring of the vertices $V_{i,1}, \dots, V_{i,m}$.

Since no two parallel segments can be adjacent, no two adjacent edges share a color. Moreover, every vertex $V_{i,j}$ is incident to exactly one edge of each color except C_j . Now, for every j , color the edge $\overline{V_0 V_{i,j}}$ the color C_j . Then, every vertex of G_i is now incident to exactly one edge of each color.

1 point for connecting them to V_0 .

For all $1 \leq i, j \leq n$, we will connect G_i and G_j in the following fashion: For all $1 \leq p, q \leq m$, we color the edge $\overline{V_{i,p}V_{j,q}}$ the color C_k , where $1 \leq k \leq m$ is the unique integer such that $p + k \equiv q \pmod{m}$. Then, every node in G_i is incident to exactly one edge of every color that connects it to either V_0 or another node in G_i , and for every other graph G_j , exactly one edge of every color connecting it to some node in G_j . Therefore, every node is now incident to exactly n edges of each color, and so we are done. \square

1 point for finishing the construction where m and n are odd.

- *Second approach.* Assume m and n are odd. Label the $mn + 1$ points

$$V_0, V_1, V_2, \dots, V_{mn}.$$

Then, for all $0 < i, j \leq mn$, color the segment $\overline{V_iV_j}$ the color C_k such that $i + j \equiv k \pmod{m}$.

2 points for coloring the edges not involving V_0 .

Furthermore, for all $0 < i \leq mn$, color $\overline{V_0V_i}$ the color C_k such that $2i \equiv k \pmod{m}$. It is easy to check that this coloring is fair, so we are done. \square

1 point for finishing.

Remark. This proof can be split into three sections: Showing that m is even and n is odd cannot be both true, showing that a fair coloring exists if n is even, and showing that a fair coloring exists if m and n are both odd. The points earned in these three sections should be added together. However, in the third section, points are accumulative. The possible marks for this problem are 0, 1, 2, 3, 4, 5, 6, 7.

6. Do there exist real numbers $a_0, a_1, \dots, a_{2018}$, with $a_0 \neq 0$, such that the roots of the polynomial $P(x) = x^{2019} + a_{2018}x^{2018} + \dots + a_1x + a_0$ are $a_0, a_1, \dots, a_{2018}$?

Solution. I claim that no such polynomial exists. We will prove a stronger result: That no such polynomial exists with $\deg P = n \geq 6$.

Suppose there exist a_0, a_1, \dots, a_{n-1} satisfying the problem statement. By Vieta's, we have that

$$\sum_{i=0}^{n-1} a_i = -a_{n-1}; \quad \sum_{i<j} a_i a_j = a_{n-2}; \quad \left| \prod_{i=0}^{n-1} a_i \right| = a_0 \Rightarrow \left| \prod_{i=1}^{n-1} a_i \right| = 1 \text{ since } a_0 \neq 0.$$

Award no points for simply stating these relationships.

Since

$$\sum_{i=0}^{n-1} a_i^2 = \left(\sum_{i=0}^{n-1} a_i \right)^2 - 2 \sum_{i<j} a_i a_j = a_{n-1}^2 - 2a_{n-2}, \quad (\star)$$

subtracting $a_{n-1}^2 + a_{n-2}^2$ from both sides we get

$$\sum_{i=0}^{n-3} a_i^2 = -a_{n-2}^2 - 2a_{n-2} = 1 - (a_{n-2} + 1)^2.$$

1 point for obtaining this relation.

By the trivial inequality,

$$\sum_{i=0}^{n-3} a_i^2 = a_0^2 + \sum_{i=1}^{n-3} a_i^2 \geq a_0^2 + 0 > 0$$

since $a_0 \neq 0$, hence we have that $1 - (a_{n-2} + 1)^2 > 0 \Rightarrow -2 < a_{n-2} < 0$. Also, for the same reason, $1 - (a_{n-2} + 1)^2 \leq 1 - 0 = 1$, hence we have that

$$\sum_{i=0}^{n-3} a_i^2 \leq 1 \Rightarrow 0 < |a_0| \leq 1, |a_i| < 1$$

for $i = 1, \dots, n-3$.

1 more point for establishing these bounds.

We can now see that $0 < -2a_{n-2} < 4$ and $0 < 1 - (a_{n-2} + 1)^2 \leq 1$. By AM-GM,

$$\sqrt[n-3]{\prod_{i=1}^{n-3} a_i^2} \leq \frac{\sum_{i=1}^{n-3} a_i^2}{n-3} < \frac{\sum_{i=0}^{n-3} a_i^2}{n-3} = \frac{1 - (a_{n-2} + 1)^2}{n-3} \Rightarrow \left| \prod_{i=1}^{n-3} a_i \right| < \left[\frac{1 - (a_{n-2} + 1)^2}{n-3} \right]^{\frac{n-3}{2}} \leq \left(\frac{1}{n-3} \right)^{\frac{n-3}{2}}.$$

By the special case of Cauchy-Schwarz inequality $\left(\sum_{i=1}^n t_i \right)^2 \leq n \sum_{i=1}^n t_i^2$, we get that

$$\frac{\sum_{i=0}^{n-2} |a_i|}{n-1} \leq \sqrt{\frac{\sum_{i=0}^{n-2} a_i^2}{n-1}} = \sqrt{\frac{-2a_{n-2}}{n-1}} \text{ by } (\star) \Rightarrow \sum_{i=0}^{n-2} |a_i| \leq \sqrt{-2a_{n-2}(n-1)} < \sqrt{4(n-1)} = 2\sqrt{n-1}.$$

3 points for bounding both $\left| \prod_{i=1}^{n-3} a_i \right|$ and $\sum_{i=0}^{n-2} |a_i|$. If only one is present, award only 1 point.

Coming back to the original Vieta's relationships we wrote at the beginning, we can now see that

$$\sum_{i=0}^{n-1} a_i = -a_{n-1} \Rightarrow |a_{n-1}| = \frac{\left| \sum_{i=0}^{n-2} a_i \right|}{2} < \frac{\sum_{i=0}^{n-2} |a_i|}{2} < \frac{2\sqrt{n-1}}{2} = \sqrt{n-1}.$$

Also, since $\left| \prod_{i=1}^{n-1} a_i \right| = 1$ and $|a_{n-2}| < 2$, we have that

$$1 = |a_{n-1}| |a_{n-2}| \left| \prod_{i=1}^{n-3} a_i \right| < \sqrt{n-1} \cdot 2 \cdot \left(\frac{1}{n-3} \right)^{\frac{n-3}{2}} = 2 \sqrt{\frac{n-1}{(n-3)^{n-3}}} \Rightarrow (n-3)^{n-3} < 4(n-1)$$

which is true only if $n < 6$, and we win. \square

2 points for arriving at the desired contradiction.

Remark. Award no points for stating that no such polynomial exists without proof. The problem requires bounding variables to reach the desired conclusion, and so points are additive. The possible marks for this problem are 0, 1, 2, 3, 5, 7.