2^{nd} Christmas Junior Mathematical Olympiad
Day 1. 270 minutes
January 4, 2019 – January 25, 2019

Note: For any geometry problems whose statement begins with an asterisk ($*$), the first page of the solution must be a large, in-scale, clearly labeled diagram. Failure to meet this requirement will result in an automatic 1-point deduction.

CJMO 1. Call a convex equilateral polygon *rhomboidal* if it can be tiled with a finite number of non-overlapping rhombi that have the same side length of the polygon. Prove that a convex equilateral polygon is rhomboidal if and only if each side of the polygon is parallel to some other side of the polygon.

CJMO 2. Prove that if a, b, c are real numbers, and the polynomial $P(x) = x^3 + ax^2 + bx + c$ has only real roots, then

$$(b-1)^2 \le \left(\frac{a^2-2b}{3}+1\right)^3,$$

and determine when equality occurs.

CJMO 3. ($*$) Let I be the incenter of $\triangle ABC$, and M be the midpoint of $\overline{BC}$. Let Ω be the nine-point circle of $\triangle BIC$. Suppose that $\overline{BC}$ intersects Ω at a point $D \neq M$. If Y is the intersection of $\overline{BC}$ and the A-intouch chord, and X is the projection of Y onto $\overline{AM}$, prove that X lies on Ω, and the intersection of the tangents to Ω at D and X lies on the A-intouch chord of $\triangle ABC$.

Note. The nine-point circle of $\triangle ABC$ is the circumcircle of its medial triangle, and if the incircle touches $\overline{AC}$ and $\overline{AB}$ at E and F, respectively, then $\overline{EF}$ is the A-intouch chord.

2^{nd} Christmas Junior Mathematical Olympiad
Day 2. 270 minutes
January 4, 2019 – January 25, 2019

Note: For any geometry problems whose statement begins with an asterisk $(*)$, the first page of the solution must be a large, in-scale, clearly labeled diagram. Failure to meet this requirement will result in an automatic 1-point deduction.

CJMO 4. $(*)$ Let ABC be a triangle with orthocenter H, and define E and F as the intersections of $\overline{AH}$ with the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ respectively. Furthermore, let D be the intersection of $\overline{BE}$ and $\overline{CF}$. Suppose that X and Y lie on $\overline{AB}$ and $\overline{AC}$ respectively such that $\overline{FX}$, $\overline{EY}$, and $\overline{BC}$ are all parallel. Prove that X and Y lie on the exterior angle bisector of $\angle BDC$.

CJMO 5. Let S be a set of $mn+1$ points equally spaced around a circle. Exactly one line segment is drawn between every pair of points in S, and each line segment is colored one of m colors. Call a coloring of line segments *fair* if for any color C of the m colors and any point P in S, P is the endpoint of exactly n line segments of color C. Find all ordered pairs of positive integers (m, n) such that a fair coloring exists.

CJMO 6. Do there exist real numbers $a_0, a_1, \ldots, a_{2018}$, with $a_0 \neq 0$, such that the roots of the polynomial $P(x) = x^{2019} + a_{2018}x^{2018} + \cdots + a_1x + a_0$ are $a_0, a_1, \ldots, a_{2018}$?

2019 Christmas Junior Mathematical Olympiad Solutions

CJMO 1 (Federico Clerici)**.**

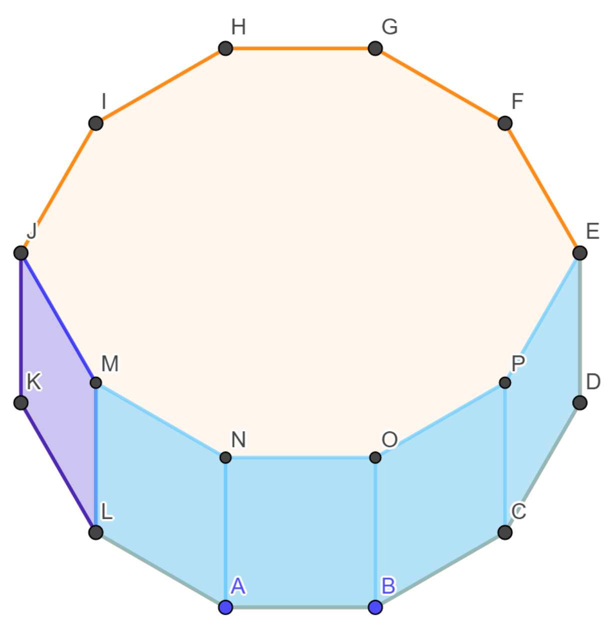
We will prove that an equilateral polygon $\mathcal{P}$ is *rhomboidal* iff for each side ℓ_i there exists another side ℓ_i' with $\ell_i \parallel \ell_i'$; this also means that $\mathcal{P}$ has an even number of sides.

- Part 1: If $\mathcal{P}$ is *rhomboidal*, for each side ℓ_i there exists another side ℓ_i' with $\ell_i \parallel \ell_i'$.

  Suppose that $\mathcal{P}$ is *rhomboidal*: each side of $\mathcal{P}$ must be a side of exactly one rhombus (if they were more than one, since the rhombi share the same side length of $\mathcal{P}$, the two or more rhombi would overlap). Let's start by placing a rhombus on the side ℓ_1 of $\mathcal{P}$, and consider the side $a_1 \parallel \ell_1$ of the rhombus: we place another rhombus on a_1, and we continue placing rhombi on the side a_n of the n^{th} rhombus, where $a_1 \parallel a_2 \parallel \ldots \parallel a_n$. Since we want to tile $\mathcal{P}$ with a finite number of rhombi, this construction should end at some point, so one of the sides $a_i \parallel \ell$ of the rhombi must be a side ℓ_1' of $\mathcal{P}$. Since this construction is valid for each side of $\mathcal{P}$, each side ℓ_i of $\mathcal{P}$ is parallel to one (and one only since $\mathcal{P}$ is convex) side ℓ_i' of the polygon. Hence our claim is proved; in particular, it follows that $\mathcal{P}$ has an even number of sides.

- Part 2: For each side ℓ_i if there exists another side ℓ_i' with $\ell_i \parallel \ell_i'$, then $\mathcal{P}$ is *rhomboidal*.

  We will prove the claim by induction. Clearly, a quadrilateral with opposite and parallel sides is a rhombus. Suppose now that all equilateral polygons $\mathcal{P}$ of $2n$ sides with each side parallel to another side of the polygon are *rhomboidal*. Consider $n+2$ consecutive vertices of an equilateral polygon $\mathcal{P}'$ with $2n+2$ sides: the first 3 such vertices identify two sides of a rhombus (see the diagram, in blue); drawing the other two sides and using the same construction of Part 1, we place other $n-2$ rhombi (in light blue), each of them with two sides parallel to the one of the first rhombus, and this tiling closes on $\mathcal{P}'$ because of our hypothesis of parallelism. Hence, we are left with an equilateral polygon with $2n$ sides, which by our inductive hypothesis, is *rhomboidal*. Hence, $\mathcal{P}'$ is *rhomboidal*, and our claim has been proven. Done. $\square$



CJMO 2 (Justin Lee, Sean Li).

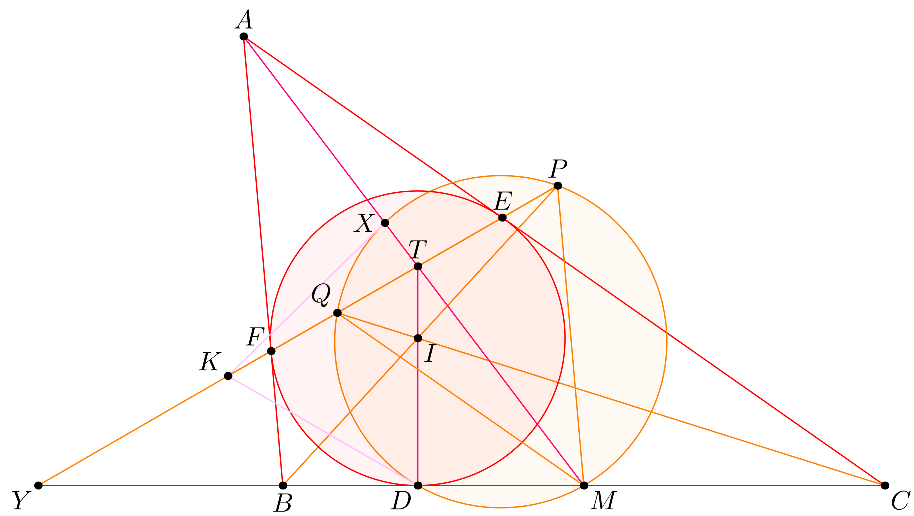
Let P's roots be r_1, r_2, r_3. Then, by AM-GM,

$$\begin{aligned}\left(\frac{a^2-2b}{3}+1\right)^3 &= \left(\frac{(r_1^2+1)+(r_2^2+1)+(r_3^2+1)}{3}\right)^3 \\ &\geq (r_1^2+1)(r_2^2+1)(r_3^2+1) \\ &= P(i)P(-i) = (c-a)^2+(b-1)^2 \geq (b-1)^2,\end{aligned}$$

where equality holds iff $a = c$ and $|r_1| = |r_2| = |r_3|$. It is not hard to check that equality holds iff $(a, b, c) = (0, 0, 0), (3\sqrt{3}, 9, 3\sqrt{3}), (-3\sqrt{3}, 9, -3\sqrt{3})$.

CJMO 3 (Eric Shen)**.**

First solution. Let the incircle of $\triangle ABC$ touch $\overline{AC}$ and $\overline{AB}$ at E and F, respectively. Furthermore, let $P = \overline{BI} \cap \overline{EF}$ and $Q = \overline{CI} \cap \overline{EF}$. By the Iran Lemma, $\angle BPC = \angle BQC = 90^\circ$, so $MP = MQ$. Let $T = \overline{AM} \cap \overline{EF}$. Obviously the incircle of $\triangle ABC$ touches $\overline{BC}$ at D.



It is well-known that T lies on $\overline{ID}$. Then, by Ceva-Menelaus,

$$-1 = (B, C; D, Y) \stackrel{I}{=} (P, Q; T, Y).$$

However, by construction, $\angle TXY = 90^\circ$, so by a well-known lemma, $\overline{XT}$ bisects $\angle PXQ$. Since $\triangle DPQ$ is the orthic triangle of $\triangle BIC$, $(DPQ) = \Omega$. However, because $MP = MQ$, M is the midpoint of $\widehat{PQ}$ in Ω. By Apollonian circles, X is unique point on $\overline{AM}$ such that $\overline{XM}$ bisects $\angle PXQ$, whence $X \in (PMQ)$. Then, notice that

$$-1 = (P, Q; T, Y) \stackrel{M}{=} (P, Q; X, D),$$

and it follows that the intersection of the tangents to Ω at D and X lies on $\overline{PQ}$, which is the A-intouch chord, as required. $\square$

Second solution. Assume WLOG $\angle B > \angle C$. Clearly D is the point where the incircle touches $\overline{BC}$. Let $\overline{EF}$ be the A-intouch chord, H be the orthocenter of $\triangle BIC$, and N and S be the midpoints of $\overline{HI}$ and $\overline{HC}$, respectively. It is well-known that $\overline{AM}, \overline{EF}, \overline{ID}$ concur at a point, say T. Since $TXYD$ is cyclic,

$$\angle MXD = \angle TXD = \angle TYD = 180 - \angle CEY - \angle YCE = 90 - \frac{A}{2} - C.$$

However, if H_I and I_A denote the reflections of H over D and M, respectively, so that they lie on the circumcircle of $\triangle BIC$. If L is the intersection of the angle bisector of $\angle BIC$ with (BIC), since $\widehat{H_I L} = \widehat{L I_A}$,

$$\angle MND = \angle I_A I H_I = 2\angle LIH_I = 2\angle CID - 2\angle LIC = 90 - \frac{A}{2} - C,$$

and $X \in \Omega$. If K is the midpoint of $\overline{YT}$ so that K is the circumcenter of $YDTX$, then

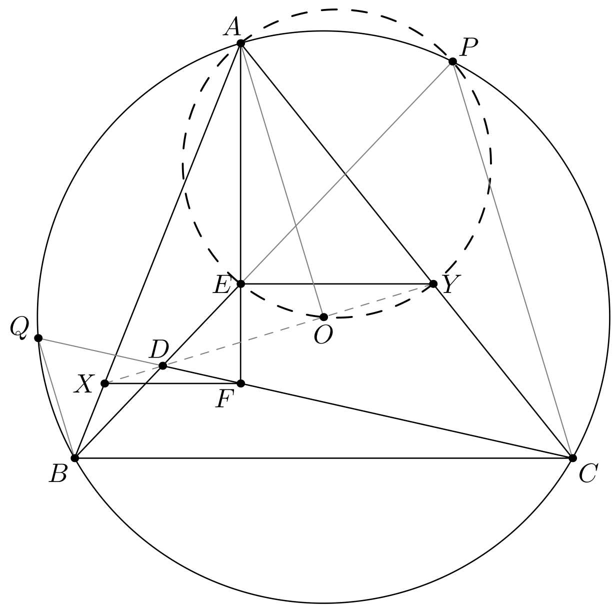
$$\measuredangle KXD = 90 - \measuredangle DYX = 90 - \measuredangle DTX = 90 - \measuredangle DTM = \measuredangle TMD = \measuredangle XMD,$$

so $\overline{KX}$ is tangent to Ω. Furthermore, $KD^2 = KX^2$, so we are done. $\square$

CJMO 4 (Kaiwen Li)**.**
Let O be the circumcenter of ABC, and define P and Q as the second intersections of $\overline{BD}$ and $\overline{CD}$ with the circumcircle of ABC. It follows that

$$\widehat{QP} = 2(\angle FCA + \angle EBA) = 2(\angle FAC + \angle EAB) = 2\angle A = \widehat{BC},$$

$BQPC$ must be an isosceles trapezoid with bases parallel to $\overline{AO}$. From here, it follows that $\overline{DO}$ is perpendicular to $\overline{AO}$ and $\overline{DO}$ bisects $\angle EDF$.



Now, let Y' be the intersection of $\overline{DO}$ with $\overline{AC}$. Trivial angle chasing yields that $\measuredangle OY'A = \measuredangle ABC = \measuredangle OEA$, so A, E, O, and Y' are concyclic and $\angle AEY' = \angle AOY' = 90°$; hence, $Y = Y'$, and $AEOY$ is cyclic. Similarly, X lies on $\overline{DO}$, and $AFOX$ is cyclic. It follows that $\overline{XY}$ is the external angle bisector of $\angle BDC$, as desired. $\square$

CJMO 5 (Joseph Zhang)**.**
The answer is all (m, n) such that at least one of the following assertions is true:

- m is odd;
- n is even.

First, we show that this is impossible if m is even and n is odd. Note that $mn + 1$ must then be odd. However, it follows that the edges of each color form a n-regular graph. Since the graph has an odd number of vertices, this is impossible by the Handshaking Lemma.

A fair coloring clearly exists if n is even. Suppose the distance $d(P, Q)$ between two points P and Q on a circle is one more the number of points between them on the shortest path around the circle. In other words, the distance between two points is the smallest number of arcs one must pass through to get between the two points going around the circle. Label the colors C_1 to C_m. Then, for each pair of points P and Q, color the segment connecting P and Q color C_i if and only if

$$\frac{n}{2}(i-1) < d(P,Q) \leq \frac{n}{2} \cdot i.$$

It is easy to see that this coloring is fair.

Now, it suffices to show that if m and n are both odd, a fair coloring exists. We have two ways to finish.

- *First approach.* From here on, assume that m and n are odd. Label the $mn + 1$ points

  $$V_0,\ V_{1,1},\ V_{1,2},\ \ldots,\ V_{1,m},\ V_{2,1},\ \ldots,\ V_{n,m}.$$

  Furthermore, for all $1 \leq i \leq n$, let G_i be the complete graph containing the points $V_{i,1}, V_{i,2}, \ldots, V_{i,m}$. We color G_i in a manner such that no two adjacent edges are the same color. Suppose that the vertices of G_i are arranged in a regular n-gon in the order $V_{i,1}, V_{i,2}, \ldots, V_{i,m}$. Now, for every side of the polygon, if the opposite vertex is $V_{i,j}$, color the side C_j. Furthermore, color every diagonal the same color as the side parallel to it.

  Since no two parallel segments can be adjacent, no two adjacent edges share a color. Moreover, every vertex $V_{i,j}$ is incident to exactly one edge of each color except C_j. Now, for every j, color the edge $\overline{V_0 V_{i,j}}$ the color C_j. Then, every vertex of G_i is now incident to exactly one edge of each color.

  For all $1 \leq i, j \leq n$, we will connect G_i and G_j in the following fashion: For all $1 \leq p, q \leq m$, we color the edge $\overline{V_{i,p} V_{j,q}}$ the color C_k, where $1 \leq k \leq m$ is the unique integer such that $p + k \equiv q \pmod{m}$. Then, every node in G_i is incident to exactly one edge of every color that connects it to either V_0 or another node in G_i, and for every other graph G_j, exactly one edge of every color connecting it to some node in G_j. Therefore, every node is now incident to exactly n edges of each color, and so we are done. □

- *Second approach.* Assume m and n are odd. Label the $mn + 1$ points

  $$V_0, V_1, V_2, \ldots, V_{mn}.$$

  Then, for all $0 < i, j \leq mn$, color the segment $\overline{V_i V_j}$ the color C_k such that $i + j \equiv k \pmod{m}$. Furthermore, for all $0 < i \leq mn$, color $\overline{V_0 V_i}$ the color C_k such that $2i \equiv k \pmod{m}$. It is easy to check that this coloring is fair, so we are done. □

CJMO 6 (Federico Clerici)**.**
I claim that no such polynomial exists. We will prove a stronger result: That no such polynomial exists with $\deg P = n \geq 6$.

Suppose there exist $a_0, a_1, \cdots, a_{n-1}$ satisfying the problem statement. By Vieta's, we have that

$$\sum_{i=0}^{n-1} a_i = -a_{n-1}; \qquad \sum_{i<j} a_i a_j = a_{n-2}; \qquad \left|\prod_{i=0}^{n-1} a_i\right| = a_0 \Rightarrow \left|\prod_{i=1}^{n-1} a_i\right| = 1 \text{ since } a_0 \neq 0.$$

Since

$$\sum_{i=0}^{n-1} a_i^2 = \left(\sum_{i=0}^{n-1} a_i\right)^2 - 2\sum_{i<j} a_i a_j = a_{n-1}^2 - 2a_{n-2}, \tag{$\star$}$$

subtracting $a_{n-1}^2 + a_{n-2}^2$ from both sides we get

$$\sum_{i=0}^{n-3} a_i^2 = -a_{n-2}^2 - 2a_{n-2} = 1 - (a_{n-2}+1)^2.$$

By the trivial inequality,

$$\sum_{i=0}^{n-3} a_i^2 = a_0^2 + \sum_{i=1}^{n-3} a_i^2 \geq a_0^2 + 0 > 0$$

since $a_0 \neq 0$, hence we have that $1 - (a_{n-2}+1)^2 > 0 \Rightarrow -2 < a_{n-2} < 0$. Also, for the same reason, $1 - (a_{n-2}+1)^2 \leq 1 - 0 = 1$, hence we have that

$$\sum_{i=0}^{n-3} a_i^2 \leq 1 \Rightarrow 0 < |a_0| \leq 1, |a_i| < 1$$

for $i = 1, \cdots, n-3$.

We can now see that $0 < -2a_{n-2} < 4$ and $0 < 1 - (a_{n-2}+1)^2 \leq 1$. By AM-GM,

$$\sqrt[n-3]{\prod_{i=1}^{n-3} a_i^2} \leq \frac{\sum_{i=1}^{n-3} a_i^2}{n-3} < \frac{\sum_{i=0}^{n-3} a_i^2}{n-3} = \frac{1-(a_{n-2}+1)^2}{n-3} \Rightarrow \left|\prod_{i=1}^{n-3} a_i\right| < \left[\frac{1-(a_{n-2}+1)^2}{n-3}\right]^{\frac{n-3}{2}} \leq \left(\frac{1}{n-3}\right)^{\frac{n-3}{2}}.$$

By the special case of Cauchy-Schwarz inequality $\left(\sum_{i=1}^{n} t_i\right)^2 \leq n\sum_{i=1}^{n} t_i^2$, we get that

$$\frac{\sum_{i=0}^{n-2} |a_i|}{n-1} \leq \sqrt{\frac{\sum_{i=0}^{n-2} a_i^2}{n-1}} = \sqrt{\frac{-2a_{n-2}}{n-1}} \text{ by } (\star) \Rightarrow \sum_{i=0}^{n-2} |a_i| \leq \sqrt{-2a_{n-2}(n-1)} < \sqrt{4(n-1)} = 2\sqrt{n-1}.$$

Coming back to the original Vieta's relationships we wrote at the beginning, we can now see that

$$\sum_{i=0}^{n-1} a_i = -a_{n-1} \Rightarrow |a_{n-1}| = \frac{\left|\sum_{i=0}^{n-2} a_i\right|}{2} < \frac{\sum_{i=0}^{n-2} |a_i|}{2} < \frac{2\sqrt{n-1}}{2} = \sqrt{n-1}.$$

Also, since $\left|\prod_{i=1}^{n-1} a_i\right| = 1$ and $|a_{n-2}| < 2$, we have that

$$1 = |a_{n-1}||a_{n-2}|\left|\prod_{i=1}^{n-3} a_i\right| < \sqrt{n-1} \cdot 2 \cdot \left(\frac{1}{n-3}\right)^{\frac{n-3}{2}} = 2\sqrt{\frac{n-1}{(n-3)^{n-3}}} \Rightarrow (n-3)^{n-3} < 4(n-1)$$

which is true only if $n < 6$, and we win. $\square$

MAC CMC

Christmas Mathematics Competitions

The problems and solutions for the 2nd Annual CJMO were prepared under the direction of:

Kyle Lee
CMC Chair
CIME Chair

Eric Shen
CJMO Chair
Solutions Director

The problems for this CJMO were written and testsolved by the members of the CMC committee:

David Altizio, Allen Baranov, Federico Clerici, Justin Lee, Kyle Lee, Kaiwen Li, Sean Li, Eric Shen, Nathan Xiong, and Joseph Zhang.

Correspondence about the problems and solutions for this CJMO and orders for any of our publications should be sent by private message on AoPS to:

AOPS12142015, eisirrational, FedeX333X, and TheUltimate123.

A complete listing of our previous competitions can be found at our website:

`https://sites.google.com/view/annualcmc/`