2^{nd} Christmas Junior Math Olympiad (Day 1)

January 4–25, 2019

CJMO 1. Call a convex equilateral polygon *rhomboidal* if it can be tiled with a finite number of non-overlapping rhombi that have the same side length of the polygon. Prove that a convex equilateral polygon is rhomboidal if and only if each side of the polygon is parallel to some other side of the polygon.

CJMO 2. Prove that if a, b, c are real numbers, and the polynomial $P(x) = x^3 + ax^2 + bx + c$ has only real roots, then

$$(b-1)^2 \leq \left(\frac{a^2 - 2b}{3} + 1\right)^3,$$

and determine when equality occurs.

CJMO 3. Let I be the incenter of $\triangle ABC$, and M be the midpoint of $\overline{BC}$. Let Ω be the nine-point circle of $\triangle BIC$. Suppose that $\overline{BC}$ intersects Ω at a point $D \neq M$. If Y is the intersection of $\overline{BC}$ and the A-intouch chord, and X is the projection of Y onto $\overline{AM}$, prove that X lies on Ω, and the intersection of the tangents to Ω at D and X lies on the A-intouch chord of $\triangle ABC$.

(The nine-point circle of $\triangle ABC$ is the circumcircle of its medial triangle, and if the incircle touches $\overline{AC}$ and $\overline{AB}$ at E and F, respectively, then $\overline{EF}$ is the A-intouch chord.)

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.

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CJMO 4. Let ABC be a triangle with orthocenter H, and define E and F as the intersections of $\overline{AH}$ with the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ respectively. Furthermore, let D be the intersection of $\overline{BE}$ and $\overline{CF}$. Suppose that X and Y lie on $\overline{AB}$ and $\overline{AC}$ respectively such that $\overline{FX}$, $\overline{EY}$, and $\overline{BC}$ are all parallel. Prove that X and Y lie on the exterior angle bisector of $\angle BDC$.

CJMO 5. Let S be a set of $mn + 1$ points equally spaced around a circle. Exactly one line segment is drawn between every pair of points in S, and each line segment is colored one of m colors. Call a coloring of line segments *fair* if for any color C of the m colors and any point P in S, P is the endpoint of exactly n line segments of color C. Find all ordered pairs of positive integers (m, n) such that a fair coloring exists.

CJMO 6. Do there exist real numbers $a_0, a_1, \ldots, a_{2018}$, with $a_0 \neq 0$, such that the roots of the polynomial $P(x) = x^{2019} + a_{2018}x^{2018} + \cdots + a_1 x + a_0$ are $a_0, a_1, \ldots, a_{2018}$?

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.