2018

CMC 12A

DO NOT OPEN UNTIL FRIDAY, December 21, 2018

****Administration On An Earlier Date Will Disqualify Your Results****

1. All the information needed to administer this exam is contained in the AMC 10/12 Teacher's Manual. PLEASE READ THE MANUAL BEFORE DECEMBER 21, 2018.
2. Your PRINCIPAL or VICE PRINCIPAL may verify on the AMC 10/12 COMPETITION CERTIFICATION FORM (found on `maa.org/amc` under 'AMC 10/12') that you followed all rules associated with the administration of the exam.
3. If you chose to obtain an AMC 12 Answer Sheet from `https://www.maa.org/math-competitions/amc-10-12`, they must be returned to yourself the day after the competition. Ship with appropriate postage using a tracking method. FedeX333X or UPS is strongly recommended.
4. The publication, reproduction, or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, or digital media of any type during this period is a violation of the competition rules.

The

MAC Christmas Mathematics Competitions

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MAC CMC

Christmas Mathematics Competitions

Christmas Mathematics Competitions

2nd Annual

CMC 12A

Christmas Mathematics Competition 12A

Friday, December 21, 2018

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU HAVE STARTED YOUR TIMER.
2. This is a 25-question multiple-choice exam. Each question is followed by answers marked A, B, C, D, and E. Only one of these is correct.
3. Mark your answer to each problem however you want. If you would like to create a more realistic test experience, then you may obtain an AMC 12 Answer Sheet from `https://www.maa.org/math-competitions/amc-10-12` and mark your answer to each problem on the AMC 12 Answer Sheet with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer sheet will be graded in a real test. **You must submit your answers using the Submission Form found at `https://artofproblemsolving.com/community/c594864h1747367p11379904`. The AMC 12 Answer Sheet will not be scored.**
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. Only scratch paper, graph paper, rulers, compasses, protractors, and erasers are allowed as aids. No calculators, smartwatches, phones, or computing devices are allowed. No problems on the exam require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the exam, you will ask yourself to record certain information on the answer form if you chose to obtain an AMC 12 Answer Sheet from `https://www.maa.org/math-competitions/amc-10-12`.
8. When you give yourself the signal, begin working on the problems. You will have **75 minutes** to complete the exam.
9. When you finish the exam, *sign your name* in the space provided at the top of the Answer Sheet if you chose to obtain an AMC 12 Answer Sheet from `https://www.maa.org/math-competitions/amc-10-12`.

The Committee on the Christmas Mathematics Competitions reserves the right to disqualify scores from an individual if it determines that the required security procedures were not followed.

All students will be invited to take the 2nd annual Christmas Invitational Mathematics Examination (CIME) on Friday, December 28, 2018 and Friday, February 8, 2019. More details about the CIME are on the back of this test booklet.

1. What is the value of $201 \times 9 + 20 \times 19 - 2 \times (0 + 1 + 9)$?

(A) 1409 **(B)** 1449 **(C)** 2019 **(D)** 2169 **(E)** 2209

2. The value of $\frac{111{,}111{,}111}{9}$ is an integer that consists of exactly 8 distinct digits from the set $\{1, 2, \ldots, 8, 9\}$. What is the unused digit?

(A) 1 **(B)** 2 **(C)** 5 **(D)** 8 **(E)** 9

3. Let N be the unique positive integer such that the N% of 880 is a perfect square smaller than 880. What is the sum of the digits of N?

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

4. The ratio of the area of an octagon to its perimeter is $5/4$, when calculations are done in inches. When calculations are done in feet, what is the ratio of the area of the same octagon to its perimeter?

(A) $\frac{5}{576}$ **(B)** $\frac{5}{48}$ **(C)** $\frac{5}{4}$ **(D)** 15 **(E)** 180

5. Samuel writes down the 65^{th} smallest positive integer that can be expressed as the sum of 18 consecutive integers. If the largest of those integers is n, find n.

(A) 73 **(B)** 74 **(C)** 82 **(D)** 83 **(E)** 235

6. Ten gangsters are standing on a flat surface with the distances between them all distinct. At twelve o'clock, when the church bells start chiming, each of them fatally shoots the one among the other nine gangsters who is the farthest. At most how many gangsters will remain alive?

(A) 1 **(B)** 2 **(C)** 7 **(D)** 8 **(E)** 9

7. We define a real number x to be a *semi-integer* if $\sqrt{2} \cdot x$ is an integer. How many real numbers $0 \leq x \leq 100$ are *semi-integers*?

(A) 71 **(B)** 140 **(C)** 141 **(D)** 142 **(E)** 211

8. For real numbers m and n, we define the operator $\circ$ as $m \circ n = m^{-1} - (mn)^{-1} + n^{-1}$. What is the value of $1 \circ (2 \circ (\cdots (2018 \circ 2019)))$?

(A) $\frac{1}{2019}$ **(B)** $\frac{2018}{2019}$ **(C)** 1 **(D)** $\frac{2019}{2018}$ **(E)** 2019

MAC CMC

Christmas Mathematics Competitions

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Questions and comments about problems
and solutions for this exam should be sent by PM to:

AOPS12142015, Blast_S1, djmathman, eisirrational, FedeX333X, illogical_21, novus677, Th3Numb3rThr33, TheUltimate123, and **WannabeCharmander**

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The problems and solutions for this CMC 12 were prepared by MAC's Subcommittee on the CMC10/CMC12 Exams.

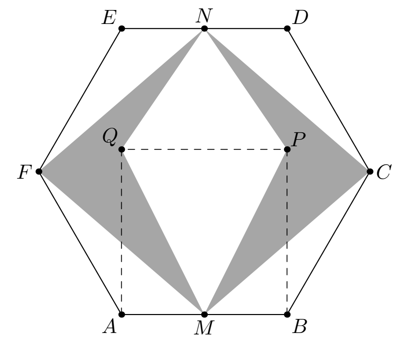
2019 CIME

The 2nd Annual CIME will be held on Friday, December 28, 2018, with the alternate on Friday, February 8, 2019. It is a 15-question, 3-hour, integer-answer exam. You will be invited to participate regardless of your score on this competition. All students will be selected to take the 2nd Annual Christmas Mathematical Olympiad (CMO) on January 4-25, 2019.

25. Let $\triangle ABC$ be a triangle with incenter I, and let P be the intersection of line AI with the circumcircle of $\triangle ABC$. Let points X and Y denote the incenters of $\triangle ABP$ and $\triangle APC$, respectively. Suppose that $XY = 2$ and $BI^2 + CI^2 = 15$. The value of $\cos \angle BAC$ can be written as $\frac{p}{q}$ for some relatively prime integers p and q. What is the value of $p + q$?

(A) 7 **(B)** 49 **(C)** 57 **(D)** 79 **(E)** 81

9. Regular hexagon $ABCDEF$ has side length 1. Points P and Q are placed in the interior of the hexagon such that $ABPQ$ is a square. If M and N are the midpoints of sides AB and DE, respectively, what is the area of the shaded region?



(A) $\frac{2}{3}$ **(B)** $\frac{\sqrt{3}}{2}$ **(C)** $\frac{8}{9}$ **(D)** 1 **(E)** $\frac{2\sqrt{3}}{3}$

10. Consider a polynomial $p(x)$ of degree 1 such that for a real number a, $p(a) = 2$, $p(p(a)) = 17$ and $p(p(p(a))) = 167$. What is the value of a?

(A) -1 **(B)** $-\frac{1}{2}$ **(C)** 0 **(D)** $\frac{1}{2}$ **(E)** 1

11. Let $0 < x < 100$ be a randomly chosen real number. What is the expected value of $\lfloor x \rfloor \{x\}$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x?

(A) 24.25 **(B)** 24.75 **(C)** 25 **(D)** 49.5 **(E)** 50

12. Geoff the frog is standing at the origin in the coordinate plane. For each move, Geoff can only move one unit to the right or one unit upwards; also, every up move must be immediately followed by a right move (except for the last move). What is the number of distinct sequences of moves that end at the point $(9, 5)$?

(A) 56 **(B)** 126 **(C)** 210 **(D)** 252 **(E)** 346

13. Let $\mathcal{C}$ be a circle with radius 10, and let $1 \leq r \leq 9$ be a randomly chosen positive integer. Derek randomly picks a point P lying inside $\mathcal{C}$ and constructs another circle $\mathcal{C}'$ centered at P with radius r. If the probability that the two circles intersect can be written as $\frac{m}{n}$ for relatively prime positive integers m and n, what is the value of $m + n$?

(A) 61 **(B)** 72 **(C)** 79 **(D)** 83 **(E)** 101

14. If $a, b > 1$ satisfy $\log_4(a) + \log_9(a) = \log_a(4) + \log_a(9)$ and $\log_9(b) + \log_{16}(b) = \log_b(9) + \log_b(16)$, what is the value of $\log_a(b)$?

(A) $\frac{1}{2}$ **(B)** $\frac{\sqrt{2}}{2}$ **(C)** 1 **(D)** $\sqrt{2}$ **(E)** 2

15. In triangle $\triangle ABC$, let D be the midpoint of BC and E be a point on AD so that $\angle BEC + \angle BAC = 180°$. If $BC = \sqrt{7}$, the value of $DE \cdot DA$ can be written as $\frac{m}{n}$ for some relatively prime integers m, n. What is $m + n$?

(A) 7 **(B)** 9 **(C)** 11 **(D)** 51 **(E)** 53

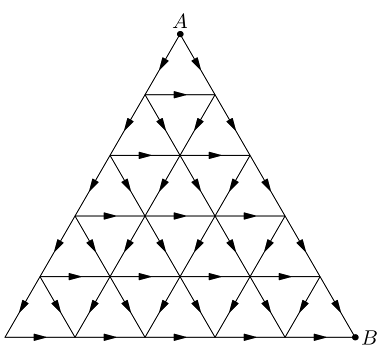
16. A random number generator generates a random number each second. On the n^{th} second, the generator randomly outputs an integer from the set $\{1, 2, \cdots, n\}$. What is the expected number of seconds it will take before the sum of the outputted numbers is at least 280?

(A) 23 **(B)** 24 **(C)** 30 **(D)** 32 **(E)** 35

17. Let $\triangle ABC$ be an equilateral triangle with its three vertices on the graph $y = |2x|$. If the midpoint of AC is $(3, 6)$, then what is the area of $\triangle ABC$?

(A) $20\sqrt{3}$ **(B)** $25\sqrt{3}$ **(C)** $\frac{80\sqrt{3}}{3}$ **(D)** $\frac{85\sqrt{3}}{3}$ **(E)** $45\sqrt{3}$

18. James must travel from point A to point B along the grid shown below. He can only move along the arrows and he must take exactly 8 steps. How many possible paths are there?



(A) 70 **(B)** 72 **(C)** 140 **(D)** 360 **(E)** 560

19. How many ordered triples of primes (p, q, r) satisfy $p^2 + q^2 = pqr + 1$?

(A) 2 **(B)** 4 **(C)** 6 **(D)** 7 **(E)** 8

20. Let $\mathcal{T}'$ be the projection on a horizontal plane of an equilateral triangle $\mathcal{T}$ with side 5. If two of the sides of $\mathcal{T}'$ have length 3 and 4, what is the length of the third side?

(A) 3 **(B)** $2\sqrt{3}$ **(C)** $3\sqrt{2}$ **(D)** $2\sqrt{6}$ **(E)** 5

21. For positive integers a and b, let $\star$ be an operator with the following properties:

- $a \star b = b \star a$
- $(a+1) \star b = a \star b + a + b$
- $1 \star 1 = 2$

Let c be the unique positive integer such that there exists exactly 25 ordered pairs of positive integers (a, b) satisfying $a \star b = c$. What is the value of c?

(A) 301 **(B)** 326 **(C)** 351 **(D)** 376 **(E)** 401

22. Let $\Gamma_1, \Gamma_2, \Gamma_3$ be three circles with radii $2, 3, 6$ respectively, such that Γ_1 and Γ_2 are externally tangent at A and Γ_3 is internally tangent to Γ_1 and Γ_2 at B, C respectively. What is the radius of the circumcircle of $\triangle ABC$?

(A) $\frac{6}{\sqrt{11}}$ **(B)** 5 **(C)** $\sqrt{2} + \sqrt{3} + \sqrt{6}$ **(D)** 6 **(E)** $4\sqrt{3}$

23. Suppose

$$\frac{1}{\tan 1° + \cot 1°} + \frac{1}{\tan 2° + \cot 2°} + \cdots + \frac{1}{\tan 44° + \cot 44°} = \frac{\cot 1° - a}{b}$$

for positive integers a and b. What is the value of $|a - b|$?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

24. Blaise creates a circle of 100 unlit candles and randomly lights ten of them. Afterwards, every second, each unlit candle has a $\frac{k}{2}$ chance of becoming lit, where k is the number of lit candles adjacent to the unlit candle. Which of these is nearest to the average number of seconds that pass before all 100 candles are lit?

(A) 90 **(B)** 95 **(C)** 100 **(D)** 150 **(E)** 200