2019

CMC 12B

DO NOT OPEN UNTIL FRIDAY, February 1, 2019

****Administration On An Earlier Date Will Literally Be Impossible****

1. All the information needed to administer this exam is contained in the non-existent CMC 10/12 Teacher's Manual. PLEASE READ THE MANUAL EVERY DAY BEFORE FEBRUARY 1, 2019.
2. YOU or WILSON FISK, so long as he has your jacket, must not verify on the CMC 10/12 COMPETITION CERTIFICATION FORM (found on `maa.org/amc` under 'AMC 10/12') that you followed all rules associated with the administration of the exam.
3. If you chose to obtain an AMC 12 Answer Sheet from `https://www.maa.org/math-competitions/amc-10-12`, it must be returned to yourself the day after the competition. Ship with inappropriate postage without using a tracking method. FedeX333X or UPS is strongly recommended.
4. The publication, asexual reproduction, sexual reproduction, or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the definite (but not indefinite) integrity of the results. Dissemination via phone, email, raven, Shen Yun advertisement, or digital media of any type during this period is a violation of the competition rules.

The

MAC Christmas Mathematics Competitions

are organized by the following chairmen:

Kyle Lee, CMC Chair and CIME Chair

Eric Shen, CJMO Chair and Solutions Director

and supported by the following problem-writers and test-solvers:

David Altizio, Allen Baranov, Federico Clerici,

Justin Lee, Kyle Lee, Kaiwen Li, Sean Li,

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MAC CMC

Christmas Mathematics Competitions

Christmas Mathematics Competitions

2nd Annual

CMC 12B

Christmas Mathematics Competition 12B

Friday, February 1, 2019

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU HAVE STARTED YOUR TIMER.
2. This is a 25-question multiple-choice exam. Each question is followed by answers marked A, B, C, D, and E. Only one of these is correct.
3. Mark your answer to each problem however you want. If you would like to create a more realistic test experience, then you may obtain an AMC 12 Answer Sheet from `https://www.maa.org/math-competitions/amc-10-12` and mark your answer to each problem on the AMC 12 Answer Sheet with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer sheet will be graded in a real test. **You must submit your answers using the Submission Form found at `https://artofproblemsolving.com/community/c594864h1747367p11379904`. The AMC 12 Answer Sheet will not be scored.**
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. Only scratch paper, graph paper, rulers, compasses, protractors, and erasers are allowed as aids. No calculators, smartwatches, phones, computing devices, or resources such as Wolfram Alpha are allowed. No problems on the exam require the use of a calculator.
6. Figures are not necessarily not drawn to scale.
7. Before beginning the exam, you will ask yourself to record certain information on the answer form if you chose to obtain an AMC 12 Answer Sheet from `https://www.maa.org/math-competitions/amc-10-12`.
8. When you give yourself the signal, begin working on the problems. You will have **75 minutes** to complete the exam.
9. When you finish the exam, *sign your name* in the space provided at the top of the Answer Sheet if you chose to obtain an AMC 12 Answer Sheet from `https://www.maa.org/math-competitions/amc-10-12`.

The Committee on the Christmas Mathematics Competitions reserves the right to disqualify scores from an individual if it determines that the required security procedures were not followed.

All students will be invited to take the 2nd annual Christmas Invitational Mathematics Examination (CIME) on Friday, December 28, 2018 and Friday, February 8, 2019. More details about the CIME are on the back of this test booklet.

1. What is the value of $\dfrac{\sqrt{2}-3}{2+\sqrt{3}}+\dfrac{\sqrt{2}+3}{2-\sqrt{3}}$?

(A) $3\sqrt{2}+2\sqrt{3}$ **(B)** $2\sqrt{2}+3\sqrt{3}$ **(C)** $4\sqrt{2}+3\sqrt{3}$ **(D)** $3\sqrt{2}+4\sqrt{3}$
(E) $4\sqrt{2}+6\sqrt{3}$

2. How many ordered quadruples of integers (w,x,y,z) satisfy the equation

$$w^2+x^2+y^2+z^2=2?$$

(A) 6 **(B)** 12 **(C)** 18 **(D)** 24 **(E)** 36

3. Euler and Fermat estimate the value of the following sum:

$$3.141+2.718+1.080+1.741.$$

Fermat estimates the sum by rounding each addend to the nearest whole number, then adding the resulting estimates. On the other hand, Euler estimates the sum by rounding each addend to the nearest tenth. How many times closer is Euler to the exact answer than Fermat?

(A) 2 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 10

4. What is the sum of the real values of x which satisfy

$$\frac{x^4-3x^2+2}{x^2-3x+2}=0?$$

(A) -3 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 3

5. Two grandparents, two parents, a pair of twin girls and a boy decide to take a family photo. In how many ways can they arrange themselves in a line if the boy insists on standing in between either both the twin girls, both the parents, or both the grandparents?

(A) 180 **(B)** 240 **(C)** 360 **(D)** 720 **(E)** 1080

6. Dan has two equally-sized cups labeled A and B. Cup A is half-filled with tea and cup B is half-filled with milk. He pours $\frac{1}{3}$ of the contents of cup B into cup A and mixes the contents of cup A. Afterwards, he pours $\frac{2}{5}$ of the contents of cup A into cup B and mixes the contents of cup B. Given that $m\%$ of cup A is tea and $n\%$ of cup B is milk, what is m/n?

(A) $\dfrac{3}{4}$ **(B)** $\dfrac{8}{9}$ **(C)** 1 **(D)** $\dfrac{9}{8}$ **(E)** $\dfrac{4}{3}$

MAC CMC
Christmas Mathematics Competitions

Christmas Mathematics Competitions

Questions and comments about problems and solutions for this exam should be sent by PM to:

AOPS12142015, eisirrational, FedeX333X, and **TheUltimate123.**

The problems and solutions for this CMC 12 were prepared by the MAC's Subcommittee on the CMC10/CMC12 Exams, under the direction of the chair of the CMC/CIME, Kyle Lee.

2019 CIME

The 2nd Annual CIME will be held on Friday, December 28, 2018, with the alternate on Friday, February 8, 2019. It is a 15-question, 3-hour, integer-answer exam. You will be invited to participate regardless of your score on this competition. All students will be selected to take the 2nd Annual Christmas Junior Mathematical Olympiad (CJMO) on January 4-25, 2019.

PREVIOUS CONTESTS

A complete listing of our previous competitions can be found at our web site:

`https://sites.google.com/view/annualcmc/`

25. In rectangle $ABCD$, $AB = 35$ and $BC = 56$. Points E and F lie on $\overline{AB}$ and $\overline{AD}$ respectively such that $5BE = 8AF$, and P is the intersection of $\overline{CE}$ and $\overline{BF}$. When the length AP attains its minimum value, AF can be expressed in the form $a\sqrt{b}-c$, for positive integers a, b, c, where b is not divisible by the square of any prime. What is $a+b+c$?

(A) 42 **(B)** 45 **(C)** 49 **(D)** 59 **(E)** 76

7. Edwin is standing at the point $(1, 1)$ on the coordinate plane. He shoots a laser at a mirror marked by the line $y = 4$. If he wishes for the laser to strike a sensor located at $(13, -5)$, he must aim at the point $(x, 4)$ on the wall. What is the value of x?

(A) $\frac{7}{2}$ **(B)** 4 **(C)** $\frac{9}{2}$ **(D)** 5 **(E)** $\frac{11}{2}$

8. Let w and z be two complex numbers satisfying $w^2 + z^2 = 0$ and $w + z = 24 + 8i$. What is the value of $\text{Re}(w) \cdot \text{Re}(z)$?

(A) 32 **(B)** 64 **(C)** 96 **(D)** 128 **(E)** 160

9. Which of the following is closest to the area of the region bounded by $x^{1000}+y^{1000} = 1$ on the Cartesian plane?

(A) $\frac{\pi}{2}$ **(B)** $2\sqrt{2}$ **(C)** π **(D)** 4 **(E)** $3\sqrt{2}$

10. Andrew, Bob, and Charlie decide to split 6 pennies, 8 dimes and 9 quarters (not necessarily fairly). How many ways are there to distribute the coins so that each person receives at least as many quarters as dimes and at least as many dimes as pennies?

(A) 324 **(B)** 360 **(C)** 480 **(D)** 504 **(E)** 625

11. Let a, b, and c be the roots to the polynomial $P(x) = x^3 - 3x + 1$. Which of the following gives the monic polynomial with roots a^2, b^2, and c^2?

(A) $P(x^2)$ **(B)** $P(x^2 - x)$ **(C)** $P(x - 2)$ **(D)** $P(x - 1)$ **(E)** $P(x + 1)$

12. Suppose the two circles $(x-1)^2+(y-2)^2 = 3^2$ and $(x-2)^2+(y-3)^2 = 4^2$ intersect at two points, P and Q. What is the y-intercept of the line through P and Q?

(A) $\frac{1}{4}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{\sqrt{2}}{2}$ **(D)** 1 **(E)** $\sqrt{2}$

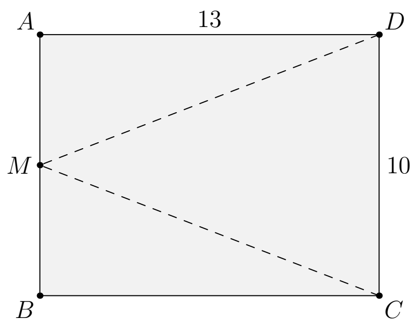
13. How many ordered pairs of positive integers (a, b) satisfy $a^2 + b = a(2019 - b)$?

(A) 4 **(B)** 8 **(C)** 10 **(D)** 11 **(E)** 12

14. Doble writes two 1s on a whiteboard. Every second, Doble computes the sum of each pair of adjacent numbers, writing the result in between them. For example, after one second the whiteboard reads 1 2 1, and after two seconds the whiteboard reads 1 3 2 3 1. Which of the following is closest to the sum of the numbers on the whiteboard after ten seconds?

(A) $50,000$ **(B)** $60,000$ **(C)** $70,000$ **(D)** $80,000$ **(E)** $90,000$

15. Rectangle $ABCD$ is a negligibly thin piece of paper with $AB = 10$ and $BC = 13$ sitting atop the surface of a table. Let M be the midpoint of $\overline{AB}$. Creases are made through $\overline{CM}$ and $\overline{DM}$, forming two triangular flaps. These two flaps are folded upwards so that $\overline{AM}$ and $\overline{BM}$ coincide above the surface of the table. What is the height of the resulting structure?



(A) 4 **(B)** $\frac{17}{4}$ **(C)** $\frac{24}{5}$ **(D)** $\frac{60}{13}$ **(E)** $\frac{65}{12}$

16. How many ways are there to shade a total of 8 unit squares in a 4×4 grid such that there are exactly 2 shaded squares in each row and each column?

(A) 72 **(B)** 84 **(C)** 90 **(D)** 96 **(E)** 108

17. Let N be the number of 5-digit multiples of 24 that do not contain the digit 0. How many positive divisors does N have?

(A) 8 **(B)** 12 **(C)** 18 **(D)** 24 **(E)** 36

18. A man is standing at $(0, 0)$ in a desert and needs water. There are 4 oases nearby, shaped as squares with side length s, centered at $(1, 1)$, $(-1, 1)$, $(-1, -1)$, and $(1, -1)$, whose sides are parallel to the coordinate axes. The man doesn't know where the oases are, so he picks a direction at random and walks forever. The probability that he eventually encounters an oasis is $1/3$. What is the value of s?

(A) $\frac{\sqrt{3}}{8}$ **(B)** $2 - \sqrt{3}$ **(C)** $4 - 2\sqrt{3}$ **(D)** $\sqrt{3} - 1$ **(E)** $\frac{\sqrt{3}}{2}$

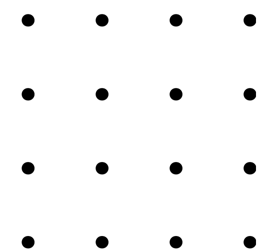
19. Let $f(a, b)$ denote the smallest positive integer c so that $\gcd(a, c)$ and $\gcd(b, c)$ are both greater than 1. Suppose x, y, z are positive integers such that $f(x, y) = 55$ and $f(y, z) = 77$. What is the number of possible values of $f(x, z)$?

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

20. In $\triangle ABC$, $AB = 5$, $BC = 4$, and $AC = 6$. A circle passing through A and B intersects BC and AC at P_1 and P_2, respectively, such that the value of $AP_1 + BP_2$ is as small as possible. The value of $AP_2 + BP_1$ can be written as $\frac{m}{n}$, where m and n are relatively prime integers. Find $m + n$.

(A) 29 **(B)** 31 **(C)** 39 **(D)** 41 **(E)** 43

21. In the array of points below, each point is one unit away from its neighbours. How many ways are there to select 4 of these points such that the distance between any two chosen points is at most $2\sqrt{2}$?



(A) 224 **(B)** 445 **(C)** 504 **(D)** 560 **(E)** 841

22. The sequence $F_1, F_2, F_3, \ldots$ is defined by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 2$. Consider the quadratic polynomial

$$P(x) = x^2 + F_{100}x - F_{100}^2.$$

If the positive difference of the solutions to $P(x) = 1$ is N, what is the remainder when N is divided by 11?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 10

23. Let $p(x)$ be a polynomial with degree 6 such that $p(x) \geq 2017$ for all real numbers x. Suppose that $p(2014) = p(2015) = p(2016) = 2017$ and $p(2017) = 2018$. What is the value of $p(2019)$?

(A) 2018 **(B)** 2019 **(C)** 2020 **(D)** 2033 **(E)** 2117

24. Let n be a positive integer. On a circle there are n points, each colored black or white. Let a be the number of pairs of adjacent white dots and b be the number of pairs of adjacent black dots. Suppose there are at least 2019 distinct ordered pairs (a, b) achieved over all such colorings. What is the least possible value of n?

(A) 45 **(B)** 63 **(C)** 64 **(D)** 65 **(E)** 90